MEMORANDUM
RM-3368-PR
JANUARY 1963

ANALYSIS OF THE RESPONSE OF MOORED SURFACE AND SUBSURFACE VESSELS TO OCEAN WAVES

J. J. Leendertse

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

MEMORANDUM
RM-3368-PR
JANUARY 1963

ANALYSIS OF THE RESPONSE OF MOORED SURFACE AND SUBSURFACE VESSELS TO OCEAN WAVES

J. J. Leendertse

This research is sponsored by the United States Air Force under Project RAND — Contract No. AF 49(638)-700 — monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Technology, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force. Permission to quote from or reproduce portions of this Memorandum must be obtained from The RAND Corporation.

The RAND Corporation
1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE

This Memorandum is part of a RAND study of water basing of weapon systems. For this study it was necessary to obtain information about the behavior of moored surface and subsurface vessels in wave-disturbed water and the induced force fluctuations in the mooring lines. This Memorandum presents an analysis of this problem.

The analysis should be of interest to defense agencies and contractors concerned with the mooring of vessels as well as to designers of moorings for various vessels including those used in the exploration and exploitation of the offshore sea bottom.

SUMMARY AND CONCLUSIONS

This study analyzes the response of a moored ship or submersible craft, in uniform or irregular waves, and also the forces in the mooring lines induced by ship responses.

Considerations are restricted mainly to waves approaching the vessel head-on, and only the surge, pitch, heave, and the fore and aft mooring-line forces are examined. Moorings with and without buoys are considered. The influence of the assumed linearization on the mooring-line force-displacement diagram for ships in irregular waves is discussed, and a method is given for estimating errors introduced by the linearization.

The response in heave, surge, and pitch of a moored ship or submersible craft with known hydrodynamic characteristics to waves approaching head-on can be calculated by use of the method outlined in this Memorandum.

Computed values of the response of a moored 880-ton vessel (simulated by a rectangular block of equivalent displacement), moored in uniform waves, are found to compare favorably with results of model tests.

High mooring-line forces are caused principally by the surge motion. These high forces can be expected when waves of significant height are present with a frequency equal to the natural frequency of the ship.

The calculation method can be expanded for the other modes of movement (sway, roll, and yaw) and for waves with arbitrary heading if the modes of movement of the unmoored ship are essentially uncoupled.

CONTENTS

LIST OF SYMBOLS

A = wave amplitude

A_s = horizontal cross-sectional area of a ship at the still water surface

A_π = horizontal cross-sectional area of a buoy at the still water surface

$\overline{A}$ = complex value of the movement in surge for a wave with unit height

a,b,c,d = coefficients in linearized mooring-line equations

B = beam

$\overline{B}$ = complex value of the movement in pitch for a wave with unit height

$\overline{C}$ = complex value of the movement in heave for a wave with unit height

d_1 = coefficient in linearized mooring-line equations

E = expectation value

F_h = resultant horizontal component of the restoring forces of the mooring cables

F_v = resultant vertical component of the restoring forces of the mooring cables

$\overline{F}^s_{ex}$ = complex value of the exciting force or moment in the s mode of movement for a wave of unit height

$f()$ = function

$\overline{f}^s_{ex}$ = complex value of the exciting force or moment per unit mass

G = spectral energy of the response for $\epsilon = o$

g = acceleration of gravity

$g(x)$ = odd-single valued power function of x

$H_{(o,o)}$ = horizontal force at the holding point (o,o)

$H_{(x,z)}$ = horizontal force at the holding point (x,z)

$H(\omega)$ = ratio of response in a particular variable to wave amplitude (complex frequency factor)

$\left[H(\omega)\right]^2$ = square of the absolute value of the complex frequency factor

h = vertical distance between the holding point of a mooring line and the sea bottom

I = imaginary axis

$I_x(t)$ = random (force) function, derived from the wave spectrum

I_θ = inertia mass moment of the ship solution around the y axis

$I_\theta(t)$ = random (moment) function, derived from the wave spectrum

I''_θ = added inertia mass moment

J_y = inertia moment of the horizontal cross-sectional area of a buoy around the y axis

j = $\sqrt{-1}$

K_{eq} = equivalent linear stiffness coefficient

$K_{s\tau}$ = stiffness coefficient in force equation of the s mode for the movement in the τ mode

k_{st} = stiffness coefficient

L = half-length of a ship

ℓ = horizontal distance between the point where a mooring line touches the sea bottom and the holding point

M_h = total moment of the horizontal components of the bow and stern lines

M_p = total moment due to the vertical forces in the mooring lines perpendicular to the long axis of the ship

$M_{s\tau}$ = virtual mass or mass inertia moment in the force on moment equation of the s mode for the movement in the τ mode

M_v = total moment of the vertical component of the bow and stern lines

M_α = expected number of maxima of the response per unit time exceeding the value of the response $R(t) = \alpha$

$M_{1_{xx}}$ = virtual mass in the x movement

$M_{2_{xx}}$ = virtual mass in the x movement

$M_{3_{xx}}$ = virtual mass in the x movement

M'' = added mass

$N_{s\tau}$ = linearized damping term in the force equation of the s mode for movement in the τ mode

n = horizontal distance between the point where a mooring line touches the sea bottom and the anchor

p = vertical distance between the holding points of a mooring line and the mass center of the ship

R_{av} = average response amplitude

R_t = periodic force due to other modes of movement

$R(t)$ = response amplitude

R_θ = restoring moment in pitch of a hovering submersible

$R_{1/3}$ = average response amplitude of the 1/3 highest responses (i.e., of the highest third of all amplitudes)

S = total length of a mooring line

$S_r(\omega)$ = spectral density of the response in a particular variable

$S_w(\omega)$ = spectral density of the waves

$\bar{S}$ = length of a mooring line lifted from the bottom

s = general indication for mode of movement

$\bar{s}$ = distance measured along a mooring line from the point where the line touches the sea bottom

T = total force in a mooring line

T_{o_s} = period of free oscillation in the s mode

t = time

$V_{(o,o)}$ = vertical force in a mooring line at the holding point o

$V_{(o,o)_p}$ = vertical component of the force in the mooring lines perpendicular to the long axis of the vessel

$V_{(x,z)}$ = vertical force in a mooring line at the holding point (x,z)

w = net weight of a mooring line per unit length

x,y,z = Cartesian co-ordinate axes

Y = random variable with zero mean value

$\bar{Z}_{ss} = -\omega^2 M_{ss} + j\omega N_{zz} + K_{zz}$ (impedance)

z_B = vertical distance between center of gravity and center of buoyancy

$\bar{z}_{ss} = \frac{1}{\omega_{o_s}^2} \frac{\bar{Z}_{ss}}{M_{ss}}$

α = a value of the response

γ = coefficient $= \frac{df(K_{eq})}{dK_{eq}}$

δ = water depth

ϵ = small parameter modifying the nonlinear function

θ = pitch angle

ζ = damping coefficient = $N/M\omega_o$

Ξ = remainder function (Eq. 85)

ρ = density of water

σ = root mean square

σ_o = root mean square of the response of the assumed linear system in surge

σ_x = root mean square of the response of the nonlinear system in surge

τ = general indication for mode of movement (used only as a subscript)

φ = angle of roll

φ_o = angle of a mooring line at the holding point with the horizontal

ψ = angle of yaw

ω = wave frequency

ω_{o_s} = natural frequency in the s mode

∇ = displacement

I. INTRODUCTION

A vessel moored at sea will experience a complicated series of translational and rotational oscillations due to sea waves. These motions can be considered as the summation of six components, three translational and three rotational.

In the presently available analyses of motions of unmoored ships and submerged crafts, differential equations can be written for each mode of movement. Unfortunately, motions in one of these modes are coupled to motions of other modes, and the analysis becomes rather complicated. Generally, the problem is simplified by neglecting some of the coupling effects and by specifying the position of the vessel in the wave system.

This study develops and analyzes a model for a moored ship or submerged craft restrained by mooring lines, using the presently available mathematical models for the free ship or hovering submerged craft and the force-displacement relationship of the cable-holding points on the ship.

The coupled movement (three degrees of freedom) in a vertical plane through the longitudinal axis of the vessel and the generated mooring-line forces are considered in detail. The general case of six degrees of freedom in arbitrary heading is discussed briefly in general terms in Section IX.

II. MOTIONS OF AN UNRESTRAINED VESSEL IN HARMONIC WAVES

Referring to the analyses by Weinblum and St. Denis,[1] the movement of a vessel unrestrained by mooring lines in harmonic waves may be expressed with certain approximations by the second-order linear differential equation

$$M_{ss} \frac{ds^2}{dt^2} + N_{ss} \frac{ds}{dt} + K_{ss} s + R_t = A\bar{F}^s_{ex} e^{j\omega t} \qquad (1)$$

where

A = wave amplitude

$\bar{F}^s_{ex}$ = wave force for a wave of unit amplitude

K_{ss} = stiffness coefficient

M_{ss} = virtual mass or mass inertia moment of the vessel

N_{ss} = linearized damping coefficient

R_t = periodic force due to the other modes of movement

s = considered translational or rotational displacement

ω = wave frequency

The first subscript of the mass, damping, and stiffness coefficients refers to the considered force or moment equation; the second subscript, to the mode of movement to which the coefficient belongs.

The first term on the left in Eq. (1) represents the inertia force; the second term represents the damping force; the third term is the restoring force; and the fourth term is the force due to other modes of movement. The term on the right expresses the periodic force of the waves.

Extensive literature is available concerning the calculations of the mass and damping coefficients for a ship of particular dimensions and the periodic wave force. Weinblum and St. Denis,[1] and Korvin-Kroukovsky,[2] particularly, present readily applicable data for calculating these coefficients and the wave forces.

Information about the coupling of the different modes of movement is limited, and only a few incidental cases have been investigated; for example, the coupled heave and pitch by Korvin-Kroukovsky and Jacobs.[3] Weinblum and St. Denis neglect the coupling in their analyses of ship motion, and in this study, the coupling term will also be neglected initially.

For the unrestrained ship, the restoring forces and moments in the different modes are caused by the displacement of the ship from the position of rest; if the ship is moored, the forces in the mooring line will, of course, cause additional restoring forces and moments.

III. MOORING-LINE CHARACTERISTICS

The forces exerted on a ship or vessel by mooring it with a long single chain or cable that has an embedded anchor at its other end are functions of the weight of the chain or line and the location of the holding point in the ship relative to the anchor. If it is assumed that the cable is lying partly on a flat bottom as in (a) of Fig. 1, then the horizontal and the vertical forces on the ship are nonlinear functions of the horizontal and vertical displacement. Based on the analyses of single mooring lines presented in Appendix A, (b) of Fig. 1 presents the total tension and its horizontal and vertical components as a function of the displacement in nondimensional parameters.

In a particular condition of the mooring chain, for example, as presented in (a) of Fig. 1, a rectangular-coordinate system is fixed to this point, with the x-axis horizontal and z-axis vertical. For small displacements around the holding point (o,o), the horizontal and the vertical components of the force in the chain at this point may be assumed to be linear with the displacement and may be expressed by

$$H_{(x,z)} = H_{(o,o)} + ax + bz \qquad (2)$$

$$V_{(x,z)} = V_{(o,o)} + cx + dz \qquad (3)$$

where

$H_{(o,o)}$ = horizontal force at the holding point o

$V_{(o,o)}$ = vertical force at the holding point o

$H_{(x,z)}$ = horizontal force at (x,z)

$V_{(x,z)}$ = vertical force at (x,z)

The coefficients a, b, c, and d can be obtained directly from Fig. 2, which is based upon the mooring-line analyses presented in Appendix A. It will be noted that $b < a$ and $c < d$.

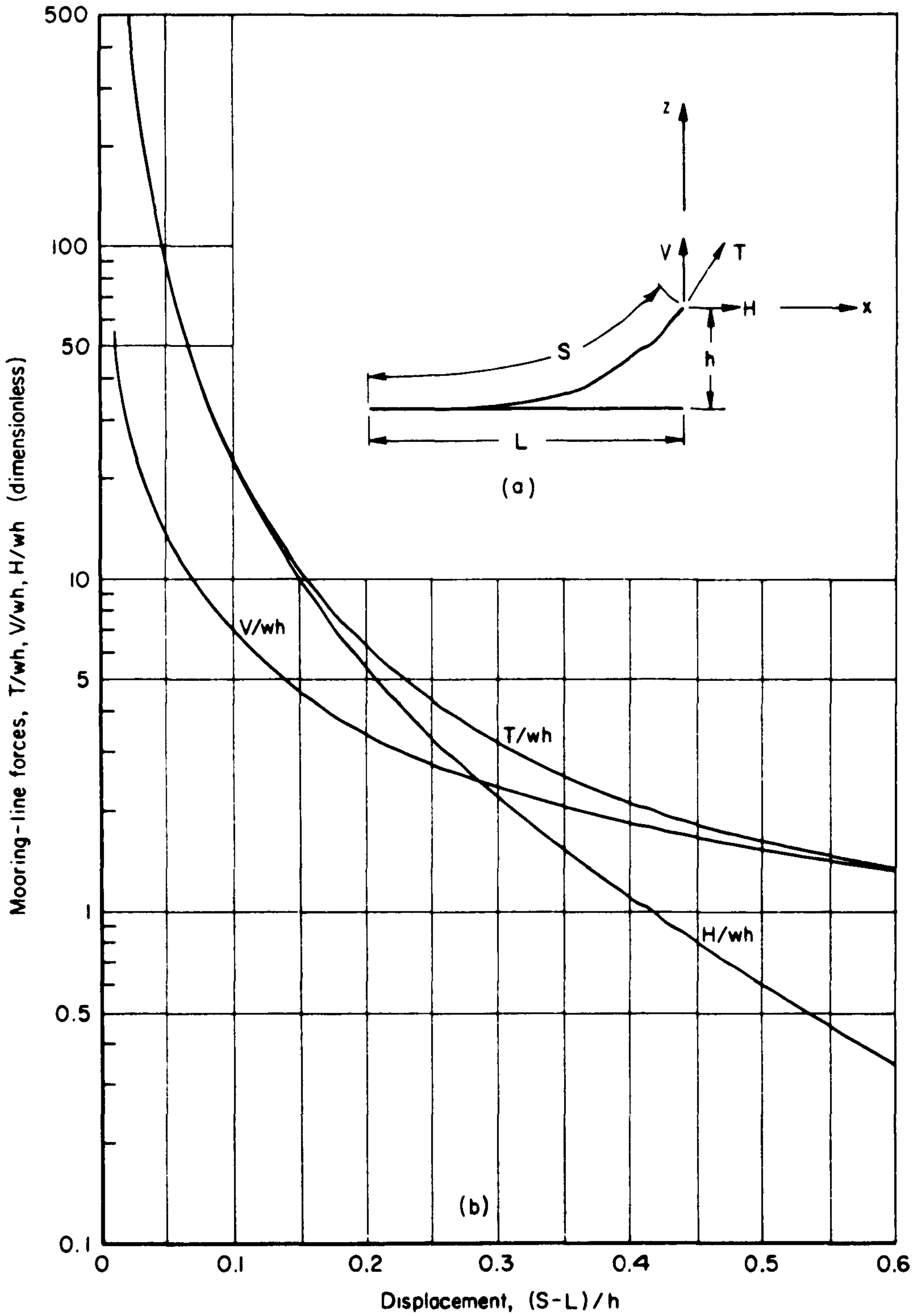


Fig. 1 — Nondimensional representation of mooring-line forces as a function of displacements

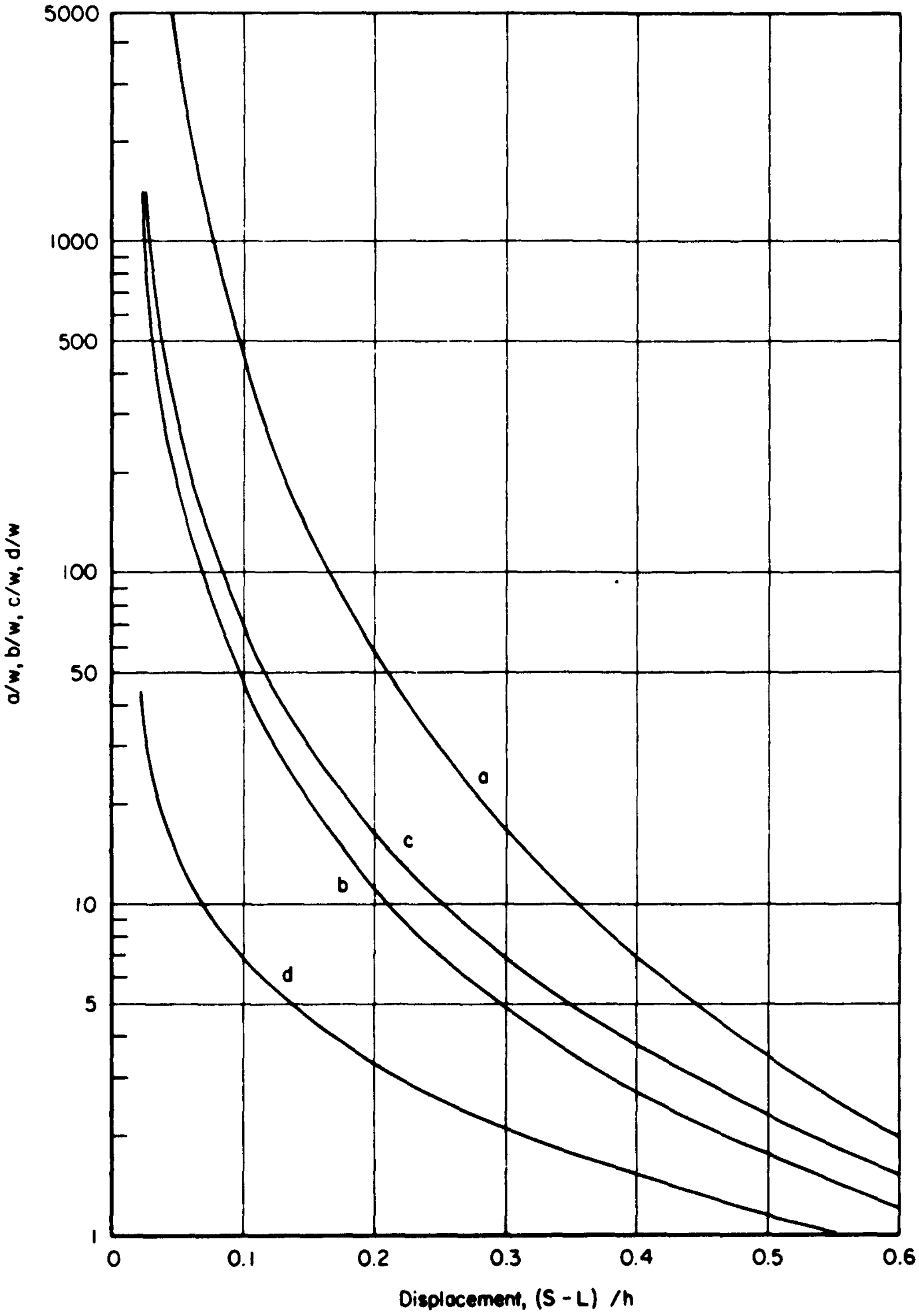


Fig. 2 — Linear coefficients for small displacement of the holding point

If a chain with a sinker is used, the forces can again be expressed by Eqs. (2) and (3), but the calculation of the coefficients becomes cumbersome.

IV. SPREAD-MOORED SHIP

Spread-mooring is used presently in the oil industry for mooring tender-barges near offshore drilling platforms. The layout of the mooring is represented in Fig. 3. It is assumed that the waves approach the ship head-on. Initially, it is assumed that the ship is subjected to uniform waves; later on, the effect of irregular waves will be introduced.

The ship's motions in the plane considered involve surging, heaving, and pitching. For the unrestrained (free-floating) ship, surge does not have important effects on the heave and pitch and consequently may be considered uncoupled. In the case of the moored ship, however, coupling will enter into the system due to the mooring lines. For example, the position of the bow, which is determined by heave and pitch, influences the horizontal component of the mooring-line force, and hence the surge.

Referring to Eq. (1), Weinblum and St. Denis,(1) and Wilson,(4) the linearized equation of motion in surge for the center of gravity of the unrestrained ship, compared to a fixed coordinate system taken in the center of the ship in still water, takes the form

$$M_{xx}\ddot{x} + N_{xx}\dot{x} = A\overline{F}^{x}_{ex}\, e^{j\omega t} \tag{4}$$

where

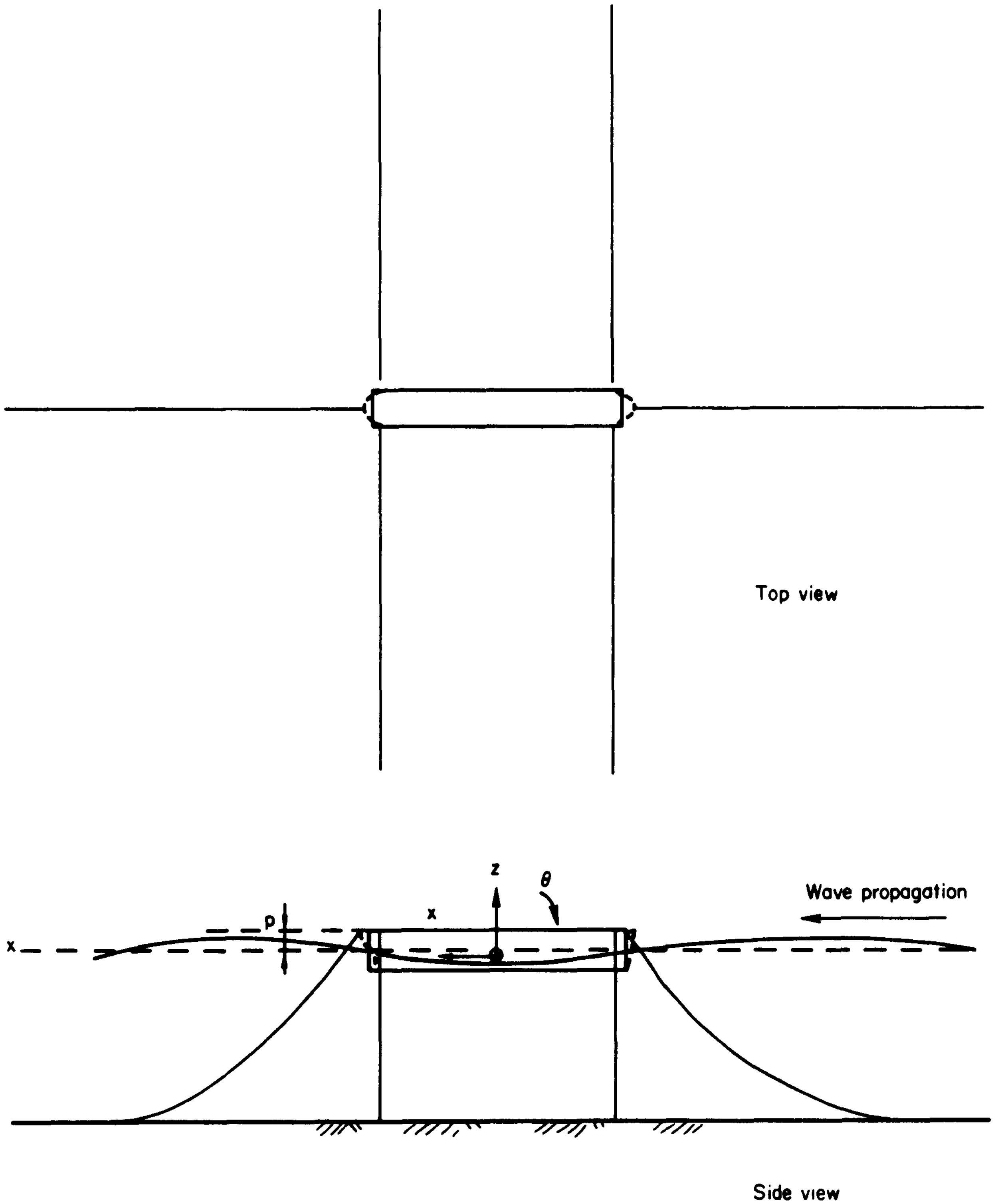
$M_{xx} = M + M''_x$ = virtual mass of the ship in x direction

M = mass of ship

M''_x = added mass in x direction

N_{xx} = damping coefficient in surge

Generally, the drag is small and may be neglected. However, in some cases, moored crafts may be built specially for mooring purposes, and in that case, no emphasis may be placed on towing or propulsion characteristics. Then, N_{xx} is not necessarily small, and estimates of values may be obtained from the propulsion characteristics and a lineari-



Note: The ship's principal axes are shown coincident with the fixed axis.

Fig. 3 — Spread-moored ship

zation process as developed by Havelock (described in Ref. 1) for the heaving motion. For the time being, the drag term will be maintained, being important even when small in cases of a resonance condition.

In addition to the inertia and drag forces, a restraining force exists for the moored vessel, and the equation of motion becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + F_h = A\bar{F}^x_{ex}\, e^{j\omega t} \tag{5}$$

where F_h is the resultant horizontal component of the restoring force of the mooring cables. With reference to Eqs. (2) and (3), taking the direction of the x-axis toward the left in the direction of wave propagation, the horizontal force of the left cable is

$$H_{(x,z)_{stern}} = H_{(o,o)_{stern}} - a(x - p\theta) + b(z + L\theta) \tag{6}$$

where

- p = vertical distance between the holding points of a mooring line and the mass center of the ship
- L = half-length of the ship

and for the cable on the upstream side

$$H_{(x,z)_{bow}} = - H_{(o,o)_{bow}} - a(x - p\theta) - b(z - L\theta) \tag{7}$$

The other four mooring lines have no significant component in the x direction. Consequently, the total restoring force is

$$- F_h = - 2ax + 2(bL + ap)\theta \tag{8}$$

Thus, Eq. (5) becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + 2ax - 2(bL + ap)\theta = \bar{F}^x_{ex}\, e^{j\omega t} \tag{9}$$

Introducing the stiffness coefficients

$$K_{xx} = 2a \tag{10}$$

and

$$K_{x\theta} = -2(bL + ap) \tag{11}$$

Eq. (9) becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{x\theta}\theta = \bar{A}\bar{F}^{x}_{ex}\, e^{j\omega t} \tag{12}$$

In this analysis, following the presentation by Kriloff,[5] Weinblum and St. Denis,[1] and Wilson,[4] the coupling effects as induced on the free-floating ship are neglected. Tests on ship models and computation of coupled and uncoupled motions indicated that neglecting the coupling terms is of minor significance for the pitching motion but is more important for the heaving motions. As will appear, since the effects of heave on the mooring-line forces are relatively minor compared with those of pitch, neglecting these coupling terms in the motion equation of the free-floating ship seems justified and simplifies the analysis significantly.

For the moored ship, the heaving motion is influenced by the restoring force of the chains. The restoring force F_v for the bow and stern chains can be calculated from the vertical mooring-line force

$$V_{(x,z)_{stern}} = -V_{(o,o)_{stern}} + c(x - p\theta) - d(z + L\theta) \tag{13}$$

$$V_{(x,z)_{bow}} = -V_{(o,o)_{bow}} - c(x + p\theta) - d(z - L\theta) \tag{14}$$

Addition of Eq. (13) and Eq. (14) gives

$$F_v = -V_{(o,o)_{stern}} - V_{(o,o)_{bow}} - 2dz \tag{15}$$

The constant forces $V_{(o,o)_{stern}} + V_{(o,o)_{bow}}$ act downward on the ship and increase its displacement. Generally, this increase is very small and may be neglected. Consequently, the stiffness coefficient in heave for the moored ship becomes

$$K_{zz} = \left(\rho g A_s + 2d\right)z \tag{16}$$

where A_s is the horizontal cross-sectional area of a ship at the still-water surface. The first term on the right side of Eq. (16) represents the vertical force due to the displaced volume of water; the second term, the force due to the mooring lines on the bow and stern of the vessel.

The coefficient d appears to be very small compared with $\rho g A_s$, and consequently the bow and stern mooring lines have an insignificant effect on the heaving motion. Likewise, the other mooring lines already neglected in Eq. (16) have no effect on the heaving motion.

Following Eq. (1), the equation of motion in pitch of a free-floating vessel may be written, if coupling with other modes of movement is neglected, as

$$M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta = A\bar{F}^{\theta}_{ex}\, e^{j\omega t} \tag{17}$$

where

$M_{\theta\theta}$ = virtual inertia moment of the ship

$N_{\theta\theta}$ = damping coefficient in pitch

$K_{\theta\theta}$ = $\rho g J_Y$

J_Y = longitudinal moment of inertia of the water plane

$\bar{F}^{\theta}_{ex}$ = exciting moment in pitch due to waves with unit height

The restoring moment $(K_{\theta\theta}\theta)$ of the free-floating ship is increased when the ship is moored.

The total moment of the vertical components of the bow and stern line is

$$M_v = -(L - p\theta)\, V_{(x,z)_{stern}} + (L + p\theta)\, V_{(x,z)_{bow}} \tag{18}$$

$$\begin{aligned} &= 2cLx - 2cpL\theta - 2dL^2\theta \\ &\quad + p\theta\left(V_{(o,o)_{stern}} + V_{(o,o)_{bow}}\right) + 2dz \end{aligned} \tag{19}$$

The moments due to the horizontal forces in the stern and bow lines are

$$M_h = -(p + L\theta)\, H_{(x,z)_{stern}} + (p - L\theta)\, H_{(x,z)_{bow}} \tag{20}$$

$$\begin{aligned} &= +2apx - 2ap^2\theta - 2pbL\theta \\ &\quad - L\theta\left(H_{(o,o)_{stern}} + H_{(o,o)_{bow}}\right) - 2bzL\theta \end{aligned} \tag{21}$$

The moments due to vertical forces in the mooring lines perpendicular to the long axis of the ship are

$$M_p = 4\left(d_1L^2 + pV_{(o,o)_p} + pd_1z\right)\theta \tag{22}$$

where

$V_{(o,o)_p}$ = vertical component of force in mooring line perpendicular to the long axis of the vessel

d_1 = coefficient determining the influence of the vertical movement

Neglecting the higher-order terms, the resultant moment due to all mooring-line forces is

$$M_h + M_v + M_p = 2(ap + cL)\,x - \Big[2ap^2 + 2(b + c)pL$$

$$+ \Big(H_{(o,o)_{stern}} + H_{(o,o)_{bow}} + 2dL + 4d_1L\Big)\,L - \Big(V_{(o,o)_{stern}} + V_{(o,o)_{bow}}$$

$$+ 4\,V_{(o,o)_p}\Big)p\Big]\,\theta \qquad (23)$$

Consequently, the total restoring moment is a function of x and θ, and the equation of motion may be written

$$M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta + K_{\theta x}x = A\bar{F}^{\theta}_{ex}\, e^{j\omega t} \qquad (24)$$

where

$$K_{\theta\theta} = \rho g J_y + \Big[2ap^2 + 2(b + c)pL + \Big(H_{(o,o)_{stern}} + H_{(o,o)_{bow}}$$

$$+ 2dL + 4d_1L\Big)\,L - \Big(V_{(o,o)_{stern}} + V_{(o,o)_{bow}} + 4\,V_{(o,o)_p}\Big)\,p\Big]$$

$$(25)$$

$$K_{\theta x} = -2(cL + ap) \qquad (26)$$

Thus, the three equations of motion are

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{x\theta}\theta = A\bar{F}^{x}_{ex}\, e^{j\omega t} \qquad (27)$$

$$M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta + K_{\theta x}x = A\bar{F}^{\theta}_{ex}\, e^{j\omega t} \qquad (28)$$

$$M_{zz}\ddot{z} + N_{zz}\dot{z} + K_{zz}z = A\bar{F}^{z}_{ex}\, e^{j\omega t} \qquad (29)$$

It will be noticed that Eqs. (27) and (28) are coupled. Anticipating a solution

$$x = \bar{A}\, e^{j\omega t} \tag{30}$$

$$\theta = \bar{B}\, e^{j\omega t} \tag{31}$$

Where $\bar{A}$ and $\bar{B}$ are complex quantities, then

$$\dot{x} = j\omega\bar{A}\, e^{j\omega t} \tag{32}$$

$$\ddot{x} = -\omega^2\, \bar{A}\, e^{j\omega t} \tag{33}$$

$$\dot{\theta} = j\omega\bar{B}\, e^{j\omega t} \tag{34}$$

$$\ddot{\theta} = -\omega^2\, \bar{B}\, e^{j\omega t} \tag{35}$$

Introducing these complex quantities in place of the real quantities in Eqs. (27) and (28) gives

$$\left(-\omega^2 M_{xx} + j\omega N_{xx} + K_{xx}\right)\bar{A} + K_{x\theta}\bar{B} = A\bar{F}^{x}_{ex} \tag{36}$$

$$K_{\theta x}\bar{A} + \left(-\omega^2 M_{\theta\theta} + j\omega N_{\theta\theta} + K_{\theta\theta}\right)\bar{B} = A\bar{F}^{\theta}_{ex} \tag{37}$$

We now introduce the impedances

$$\bar{Z}_{xx} = -\omega^2 M_{xx} + j\omega N_{xx} + K_{xx} \tag{38}$$

and

$$\bar{Z}_{\theta\theta} = -\omega^2 M_{\theta\theta} + j\omega N_{\theta\theta} + K_{\theta\theta} \tag{39}$$

which simplifies Eqs. (36) and (37) to

$$\bar{Z}_{xx}\bar{A} + K_{x\theta}\bar{B} = A\bar{F}^{x}_{ex} \tag{40}$$

$$K_{\theta x}\bar{A} + \bar{Z}_{\theta\theta}\bar{B} = A\bar{F}^{\theta}_{ex} \tag{41}$$

Solving for $\bar{A}$ and $\bar{B}$ gives

$$\bar{A} = \frac{\begin{vmatrix} \bar{F}^{x}_{ex} & K_{x\theta} \\ \bar{F}^{\theta}_{ex} & \bar{Z}_{\theta\theta} \end{vmatrix}}{\begin{vmatrix} \bar{Z}_{xx} & K_{x\theta} \\ K_{\theta x} & \bar{Z}_{\theta\theta} \end{vmatrix}} A \tag{42}$$

and

$$\bar{B} = \frac{\begin{vmatrix} \bar{Z}_{xx} & \bar{F}^{x}_{ex} \\ K_{\theta x} & \bar{F}^{\theta}_{ex} \end{vmatrix}}{\begin{vmatrix} \bar{Z}_{xx} & K_{x\theta} \\ K_{\theta x} & \bar{Z}_{\theta\theta} \end{vmatrix}} A \tag{43}$$

by which amplitudes and phase lags with the exciting periodic waves can be calculated.

For the vertical motion, a complex solution is anticipated

$$z = \bar{C}e^{j\omega t} \tag{44}$$

where $\overline{C}$ is a complex quantity. Following the method for x and θ, we obtain

$$\overline{Z}_{zz}\overline{C} = A\overline{F}^{z}_{ex} \tag{45}$$

where

$$\overline{Z}_{zz} = -\omega^2 M_{zz} + j\omega N_{zz} + K_{zz} \tag{46}$$

Thus

$$\overline{C} = \frac{\overline{F}^{z}_{ex}}{\overline{Z}_{z}} A \tag{47}$$

The fluctuations in the mooring cables may now be determined. For example, rewriting Eq. (6)

$$H_{(x,z)_{stern}} = H_{(o,o)_{stern}} - ax + (ap + bL)\theta + bz$$

If we introduce the following expression for the force fluctuation in the mooring cable, which is a function of wave amplitude and frequency

$$H_{stern}(A,\omega) = -ax + (ap + bL)\theta + bz \tag{48}$$

then

$$H_{stern}(A,\omega) = Re\left[-a\overline{A} + (ap + bL)\overline{B} + b\overline{C}\right] Ae^{j\omega t} \tag{49}$$

In many instances, the term $K_{x\theta}$ in Eq. (40) is very small compared to $\overline{Z}_{xx}$, and $K_{\theta x}$ in Eq. (41) is very small compared to $\overline{Z}_{\theta\theta}$. Then the pitch and surge of the moored ship are essentially uncoupled, and

$$\overline{A} \approx \frac{\overline{F}^{x}_{ex}}{\overline{Z}_{xx}} A \tag{50}$$

$$\bar{B} \approx \frac{\bar{F}^{\theta}_{ex}}{\bar{Z}_{\theta\theta}} \qquad (51)$$

The coupling is important, however, for the resonance movement in surge, which is generally not significantly damped, and in that case Eqs. (42) and (43) have to be used.

If coupled motion for the free-floating ship in pitch and heave are important--for example, for a ship with the center of mass not approximately in the middle of the ship as described by Korvin-Kroukovsky[2]--the equation of motion of this vessel when moored becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{x\theta}\theta = A\bar{F}^{x}_{ex}\, e^{j\omega t} \qquad (52)$$

$$K_{\theta x}x + M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta + M_{\theta z}\ddot{z} + N_{\theta z}\dot{z} + K_{\theta z}z = A\bar{F}^{\theta}_{ex}\, e^{j\omega t} \qquad (53)$$

$$M_{z\theta}\ddot{\theta} + N_{z\theta}\dot{\theta} + K_{z\theta}\theta + M_{zz}\ddot{z} + N_{zz}\dot{z} + K_{zz}z = A\bar{F}^{z}_{ex}\, e^{j\omega t} \qquad (54)$$

or using the mechanical impedances $\bar{Z}_{\theta z}$ and $\bar{Z}_{z\theta}$, similarly $\bar{Z}_{xx}$ as in Eq. (38) the equations of motion may be expressed:

$$\bar{Z}_{xx}\,\bar{A} + K_{x\theta}\,\bar{B} = A\bar{F}^{x}_{ex} \qquad (55)$$

$$K_{\theta x}\,\bar{A} + \bar{Z}_{\theta\theta}\,\bar{B} + \bar{Z}_{\theta z}\,\bar{C} = A\bar{F}^{\theta}_{ex} \qquad (56)$$

$$\bar{Z}_{z\theta}\,\bar{B} + \bar{Z}_{zz}\,\bar{C} = A\bar{F}^{z}_{ex} \qquad (57)$$

This set of linear equations may be solved by using Cramer's rule, writing for the determinant of the system

$$\Delta = \begin{vmatrix} \bar{Z}_{xx} & K_{x\theta} & 0 \\ K_{\theta x} & \bar{Z}_{\theta\theta} & \bar{Z}_{\theta z} \\ 0 & \bar{Z}_{z\theta} & \bar{Z}_{zz} \end{vmatrix} \qquad (58)$$

The unique solutions are given by

$$\bar{A} = \frac{\Delta_x}{\Delta} \tag{59}$$

$$\bar{B} = \frac{\Delta_\theta}{\Delta} \tag{60}$$

$$\bar{C} = \frac{\Delta_z}{\Delta} \tag{61}$$

where Δ_x, Δ_θ, Δ_z are the determinant forms obtained by replacing the elements of the first, second, or third columns, respectively, of Eq. (58) by $A\bar{\bar{F}}^x_{ex}$, $A\bar{\bar{F}}^\theta_{ex}$, $A\bar{\bar{F}}^z_{ex}$.

V. SHIP MOORED BY BUOYS WITH UNIFORM WAVES HEAD-ON

The equations of motion for a ship using mooring buoys can be derived in a fashion similar to that for the ship using mooring cables only. In this case, the motions of the buoys have to be considered in addition to the motions of the ship.

Considering a mooring configuration in Fig. 4, it will be noted that the relative vertical motions between the buoys and the ship will induce small horizontal displacements between the buoys and the ship, thus relatively small force fluctuations in the lines between ship and buoy. Consequently, the heaving and pitching motions of the ship are considered as of no importance to the forces in these lines. This is naturally not the case for the heaving motions of the buoys.

Assuming again a linear relationship between forces and movements, and neglecting the pitching of the buoys, the equations of motion of the system neglecting damping in surge become

$$M_{1_{xx}} \ddot{x}_1 + N_{1_{xx}} \dot{x} + a\, x_1 + K_1 (x_1 - x_2) - b\, z_1 = \bar{F}^x_{1_{ex}} e^{j\omega t} \tag{62}$$

$$M_{2_{xx}} \ddot{x}_2 + K_1 (x_2 - x_1) + K_2 (x_2 - x_3) = \bar{F}^x_{2_{ex}} e^{j\omega t} \tag{63}$$

$$M_{3_{xx}} \ddot{x}_3 + N_{3_{xx}} \dot{x} + K_2 (x_3 - x_2) + a\, x_3 + b\, z_3 = \bar{F}^x_{3_{ex}} e^{j\omega t} \tag{64}$$

$$M_{1_{zz}} \ddot{z}_1 + N_{1_{zz}} \dot{z} + \left[\rho g\, (2\, L_1 B_1) + d\right] z_1 - c x_1 = \bar{F}^z_{1_{ex}} e^{j\omega t} \tag{65}$$

$$M_{3_{zz}} \ddot{z}_3 + N_{3_{zz}} \dot{z} + \left[\rho g\, (2\, L_3 B_3) + d\right] z_3 + c x_3 = \bar{F}^z_{2_{ex}} e^{j\omega t} \tag{66}$$

In many instances, in mooring with buoys, the connection between the ship and buoy is made with a cable that is relatively light in comparison with the heavy chains used on the buoys. If these cables are placed in high tension, the horizontal movements of the buoys and the ship are practically the same, and it may be assumed that $x_1 = x_2 = x_3$.

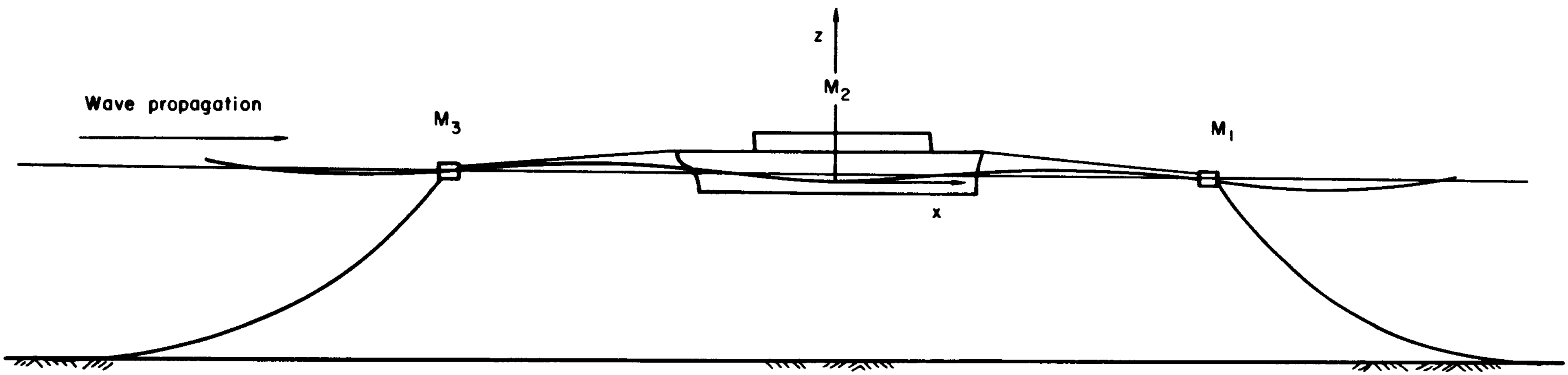


Fig. 4 — Buoy mooring

Then Eqs. (62) through (66) reduce to

$$\left(M_{1_{xx}} + M_{2_{xx}} + M_{3_{xx}}\right)\ddot{x} + \left(N_{1_{xx}} + N_{2_{xx}}\right)\dot{x} + 2ax - bz_1 + bz_3$$
$$= \left(\bar{F}^x_{1_{ex}} + \bar{F}^x_{2_{ex}} + \bar{F}^x_{3_{ex}}\right)e^{j\omega t} \qquad (67)$$

$$M_{1_{xx}}\ddot{z}_1 + N_{1_{zz}}\dot{z}_1 + \left[\rho g(A_\pi) + d\right] z_1 - cx = \bar{F}^z_{1_{ex}} \; e^{j\omega t} \qquad (68)$$

$$M_{3_{zz}}\ddot{z}_3 + N_{3_{zz}}\dot{z}_3 + \left[\rho g(A_\pi) + d\right] z_3 + cx = \bar{F}^z_{2_{ex}} \; e^{j\omega t} \qquad (69)$$

In Eq. (67), the virtual masses of the buoys are small compared with the mass of the ship, and also the horizontal wave forces are small compared with the wave force acting on the ship; consequently, the effects of the buoys in this horizontal movement of the ship may be neglected.

Generally, the natural frequency in heave of the buoys is higher than the frequencies of the waves, thus the terms $M\ddot{z}$ and $N\dot{z}$ are small compared with the term $\left[\rho g(2LB) + d\right]$ and may be neglected in our initial investigation of the ship's movement.

Disregarding the above-mentioned terms, introduction of Eqs. (68) and (69) into Eq. (67) gives

$$M_{2_{xx}}\ddot{x} + \left(N_{1_{xx}} + N_{2_{xx}}\right)\dot{x} + \left\{2a - \frac{2bc}{(\rho g A_\pi + d)}\right\} x = \left(+ \frac{b}{(\rho g A_\pi + d)}\bar{F}^z_{1_{ex}}\right.$$
$$\left. - \frac{b}{(\rho g A_\pi + d)}\bar{F}^z_{2_{ex}} + \bar{F}^x_{2_{ex}}\right) e^{j\omega t} \qquad (70)$$

This result is important, since in principle it enables the design of a mooring in which, at the resonance frequency

$$\omega = \left[\frac{2a - \dfrac{2bc}{\rho g A_\pi + d}}{M_{2_{xx}}}\right] \qquad (71)$$

the excitation term on the right side of Eq. (70) becomes zero by proper placement of the buoys.

VI. SUBMERGED CRAFT WITH UNIFORM WAVES HEAD-ON

Equations (27) through (29) developed in Section IV are also applicable to submerged craft having the configuration shown in Fig. 5. Here the vessel is assumed to be moored with two lines, one at the bow and one at the stern, and to be situated in water of intermediate depth.

Since displacements in heave do not induce changes in the volumes of water displaced by the vessel, as is the case with surface ships, the restoring forces in the equation for heave (Eq. 16) are determined only by the effect of the mooring lines. Thus

$$K_{zz} = 2dz$$

The restoring moment in pitch (R_θ) of the hovering submersible is caused by the buoyance

$$R_\theta = \rho g \nabla z_B \theta \tag{73}$$

where

z_B = distance between center of buoyancy and center of gravity

∇ = displacement

Consequently, the stiffness coefficient in pitch for the moored submersible is

$$K_{\theta\theta} = 2ap^2 + 2(b + c)pL + \left(2dL + H_{(o,o)_{stern}} + H_{(o,o)_{bow}}\right) L - \left(V_{(o,o)_{stern}} + V_{(o,o)_{bow}}\right) p + \varsigma g \nabla z_B \tag{74}$$

Goodman and Sargent,[7] studying the response of a submerged hovering submarine, neglect the damping term in the equation of motion.

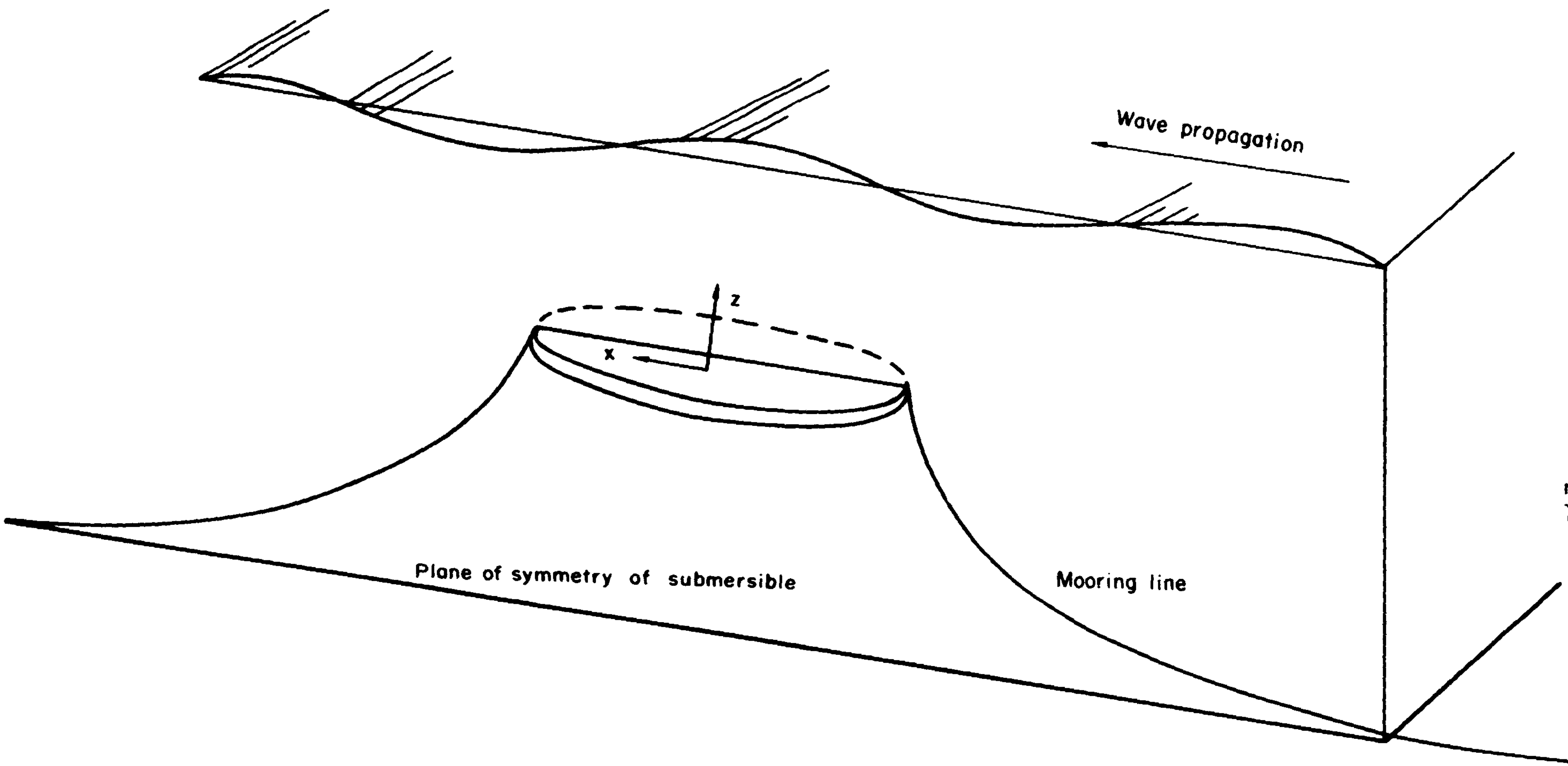


Fig. 5 — Moored submersible

Considering a submarine with approximately symmetrical fore and aft end, they obtained in the nomenclature used here

$$M_{zz}\ddot{z} = A\bar{F}^{z}_{ex}e^{j\omega t} \tag{75}$$

$$M_{\theta\theta}\ddot{\theta} + \zeta g\nabla \cdot z_{B}\theta = A\bar{F}^{\theta}_{ex}e^{j\omega t} \tag{76}$$

For the moored submersible the equations become

$$M_{xx}\ddot{x} + K_{xx}x + K_{x\theta}\theta = A\bar{F}^{x}_{ex}e^{j\omega t} \tag{77}$$

$$M_{\theta\theta}\ddot{\theta} + K_{\theta\theta}\theta + K_{\theta x}x = A\bar{F}^{\theta}_{ex}e^{j\omega t} \tag{78}$$

$$M_{zz}\ddot{z} + K_{zz}z = A\bar{F}^{z}_{ex}e^{j\omega t} \tag{79}$$

Neglecting the damping terms is justified here if no motions with frequency close to one of the natural frequencies are considered.

VII. SPREAD-MOORED SHIP AND SUBMERSIBLE IN LONG-CRESTED IRREGULAR WAVES

It appears that the actual wave condition in the ocean can best be represented by use of the model of a random process as derived by Neumann and described by Pierson, Neumann, and James.[8] Statistical values such as average wave height are given, not values of the environment as a function of time. The sea is taken as a summation of a large number (or as an integral of an infinite number) of uniform wave trains, each with different amplitudes and directions superimposed in random phase. The profiles of the individual waves are assumed to be sine curves according to Airy's Theory.[9]

Techniques are available to predict the amplitudes of the waves and their distribution over the frequency range from wind velocity, wind duration, and the fetch. Generally, the result can be presented in the form of a wave spectrum, which is the distribution of the mean squares of the wave amplitudes in a given increment of the frequency (spectral density) over the wave frequencies.

In the following analysis, it is assumed that the waves are unidirectional and meet the ship or submerged vessel head on. This case is realistic, as it represents the crafts moored in swell.

Following the work by St. Denis and Pierson,[10] the relation between the spectral density of wave and ship responses is given by

$$S_r(\omega) = S_w(\omega)\left[H(\omega)\right]^2 \tag{80}$$

where

$S_r(\omega)$ = spectral density of the response in a particular variable (displacement, strain, etc.)

$S_w(\omega)$ = spectral density of the wave

$H(\omega)$ = ratio of response in a particular variable to wave amplitude (complex frequency factor)

If the spectrum of the waves is given, the spectrum of the response can be calculated by Eq. (70). The mean square of the response is then given by

$$\sigma^2 = \int_0^{\omega} S_r(\omega)d\omega = \int_0^{\omega} S_w(\omega)\left[H(\omega)\right]^2 d\omega \tag{81}$$

It has been shown by Longuet-Higgins,(11) that for a relatively narrow band of wave frequencies, such as is the case with swells being assumed here, the probability distribution of the wave amplitudes tends to be Gaussian if the frequency factor has nonzero values in the range of wave frequencies. Consequently, it may be expected that the probability distribution of the response amplitudes is also Gaussian.

Longuet-Higgins calculated important statistical relationships for the narrow-frequency spectrum, which were consequently tabulated by Pierson, Neumann, and James;(8) for example

R_{av}	$= 0.88\ \sigma$	(Average response amplitude)
$R_{1/3}$	$= 1.416\ \sigma$	(Average response amplitude of the 1/3 highest responses)

In many instances, the response spectrum may not be considered to be narrow, and the expected number (M_α) of maxima of the response per unit time exceeding the value of the response $R(t) = \alpha$ can be expressed after Bendat(12) as

$$M_\alpha = \frac{1}{2\pi}\left(\frac{E\left[R^2(t)\right]}{\sigma^2}\right)^{1/2} e^{\left(\frac{-\alpha^2}{2\sigma^2}\right)} \tag{81a}$$

where

$$E\left[R^2(t)\right] = \int_0^{\infty} \omega^2 S_w \left[H(\omega)\right]^2 d\omega \tag{81b}$$

Thus, this presentation introduces the probability concept into the calculation of movements and cable stresses.

VIII. EFFECT OF THE NONLINEAR MOORING-LINE FORCES

In the analyses of the response, it has been assumed that the restoring forces of the cables are linear with the displacement by use of Eqs. (2) and (3). This assumption will introduce certain errors in the calculated response and the mooring-line forces.

Considering the spread-moored ship, it has been seen that the pitch and surge are coupled because of the bow and stern mooring lines.

If the total horizontal restoring force of a system is plotted as a function of the horizontal displacement for different pitch angles, a graph of the type presented in Fig. 6 will be obtained. In this graph the linearization calculated by Eqs. (2) and (3) is also plotted.

The nonlinearity of the total restoring force is much smaller than that of the individual cables.

It will be seen from such graphs that force-displacement curves for different pitch angles are essentially parallel for equal distances over the expected range of pitch angles.

It is assumed that movements in surge extend into the nonlinear range. The horizontal restoring force may now be written, following the procedures of Crandall,[13] by extending the linear Eq. (8)

$$F_h = 2a\left[x + \epsilon g(x)\right] - 2(bL + ap)\theta \qquad (82)$$

where

ϵ = small parameter modifying the nonlinear function

$g(x)$ = odd single-valued power function of x

The values ϵ and $g(x)$ are chosen in such a manner that for zero pitch angle, Eq. (82) is identical with the force-displacement curve obtained by use of Appendix A.

The coupled equations of motion in surge and pitch for the ship in irregular waves can now be written by introducing the nonlinearity in Eq. (27).

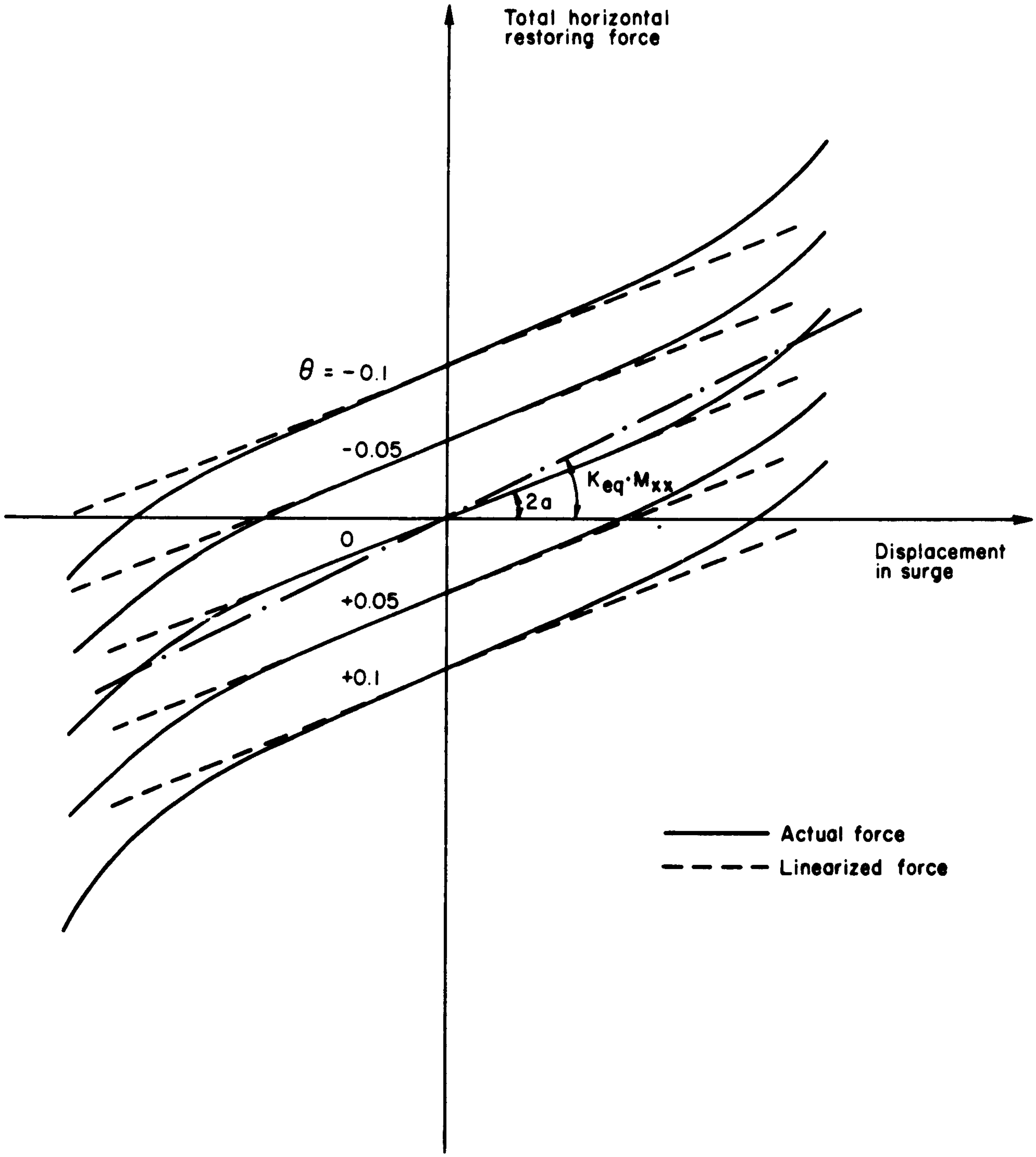


Fig. 6 — Typical plot for the horizontal restoring force versus displacement for different pitch angles

$$\ddot{x} + \frac{N_{xx}}{M_{xx}}\dot{x} + \frac{2a}{M_{xx}}\left[x + \epsilon g(x)\right] + \frac{K_{x\theta}\theta}{M_{xx}} = I_x(t) \tag{83}$$

$$\ddot{\theta} + \frac{N_{\theta\theta}}{M_{\theta\theta}}\dot{\theta} + \frac{K_{\theta\theta}}{M_{\theta\theta}}\theta + \frac{K_{\theta x}\theta}{M_{\theta\theta}} = I_\theta(t) \tag{84}$$

where $I_x(t)$ and $I_\theta(t)$ are random functions, both derived from the wave spectrum.

Equation (83) may be rewritten by introducing the equivalent linear stiffness coefficient K_{eq}

$$\ddot{x} + 2\alpha\dot{x} + K_{eq}x + k_{x\theta}\theta = I_x(t) + \Xi \tag{85}$$

where

$$\Xi = \left(K_{eq} - \omega_{o_x}^2\right)x - \epsilon\omega_{o_x}^2 g(x) \tag{86}$$

$$\omega_{o_x}^2 = 2a/M_{xx} \tag{87}$$

Assuming that Ξ is zero, the mean square response of the system to an irregular sea with a particular spectrum is found by Eq. (81)

$$\bar{\sigma}_x^2 = \int_0^\infty S_w(\omega)\left[H(\omega)\right]^2 d\omega \tag{88}$$

The spectral density $S_w(\omega)$ is given from the assumed sea condition, and the square of the absolute value of the complex-frequency factor $\left[H(\omega)\right]^2$ is obtained from Eq. (42)

$$\left[H(\omega)\right]^2 = \left| \frac{\begin{vmatrix} \bar{F}_{ex}^x & K_{x\theta} \\ \bar{F}_{ex}^\theta & \bar{Z}_{\theta\theta} \end{vmatrix}}{\begin{vmatrix} \bar{Z}_{xx} & K_{x\theta} \\ K_{\theta x} & \bar{Z}_{\theta\theta} \end{vmatrix}} \right|^2 \tag{89}$$

where

$$\bar{Z}_{xx} = -\omega^2 M_{xx} + j\omega N_{xx} + M_{xx} K_{eq} \tag{90}$$

Introducing Eqs. (89) and (90) into Eq. (88)

$$\bar{\sigma}_x^2 = G\, f\left(K_{eq}\right) \tag{91}$$

for small variation of K_{eq} from $\omega_{o_x}^2$, Eq. (91) may be expressed

$$\bar{\sigma}_x^2 = G_w \left[1 + \gamma\left(K_{eq} - \omega_{o_x}^2\right)\right] \tag{92}$$

where $G_w = \bar{\sigma}_o^2$ = spectral energy of the response for $\epsilon = o$

$$\gamma = \frac{d\, f\left(K_{eq}\right)}{d\, K_{eq}} \text{ at } K_{eq} = \omega_{o_x}^2 \tag{93}$$

In the analysis with Eqs. (88) through (92) it was assumed that the remainder function Ξ equals zero, which is naturally not the case; Ξ is again a stationary random process just like $I_x(t)$ and depends on the value of the equivalent stiffness coefficient. A measure of its value is its expected mean square $E\left[\Xi^2\right]$.

The mean square of the remainder function Ξ can be expressed by use of Eq. (86)

$$E\left[\Xi^2\right] = K_{eq}^2\, E\left[x^2\right] - 2K_{eq}\omega_o^2 E\left[x^2 + \epsilon x g(x)\right] + \omega_o^4 E\left[\left\{x + \epsilon g(x)\right\}^2\right] \tag{94}$$

This will be a minimum for fixing K_{eq} when

$$\frac{d\left(E\left[\Xi^2\right]\right)}{dK_{eq}} = 0 \tag{95}$$

which results in

$$K_{eq} = \omega_{o_x}^2 \left(1 + \epsilon \frac{E\left[xg(x)\right]}{E\left[x^2\right]}\right) \tag{96}$$

Inserting Eq. (96) into Eq. (92) results

$$\frac{\bar{\sigma}_x^2}{\bar{\sigma}_o^2} = 1 + \gamma\omega_{o_x}^2 \epsilon \frac{E\left[xg(x)\right]}{E\left[x^2\right]} \tag{97}$$

The probability density of a random variable Y with zero mean value is

$$f(Y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-Y^2/2\sigma^2} \tag{98}$$

where σ = standard deviation.

The expectation value $E\left[xg(x)\right]$ in Eq. (97) is for the nonlinear system, which would require knowing the response of the nonlinear system. Fortunately, the term in Eq. (97) is to be multiplied by the small parameter ϵ, and the expectation value $E\left[xgx\right]$ of the linear system instead of the nonlinear system will induce errors of the second order.

Consequently

$$\frac{\bar{\sigma}_x^2}{\bar{\sigma}_o^2} = 1 + \frac{\gamma\epsilon\omega_{o_x}^2}{\sqrt{2\pi}\,\bar{\sigma}_o^3} \int_{-\infty}^{+\infty} xg(x) e^{-x^2/2\sigma_o^2}\, dx \tag{99}$$

by which the effect of the linearization can be investigated. The term γ may be positive or negative.

IX. DISCUSSION

The mathematical models presented here have shortcomings. The most important one is the assumed linear relationship between the restoring forces and the displacement of the ship. The effect of the nonlinearity of the mooring lines in the surge motion, which is particularly affected by the nonlinearity, was investigated in detail in Section VIII, and a method was presented for calculating the ratio of the mean square of the nonlinear response and the linear response.

Naturally, the methods of analyzing the response of the moored ship has the limitations that are imposed on the analysis of a free-floating vessel, and the direct force-displacement relationship established in Section II limits the method to mooring in a few hundred feet depth.

Unfortunately, no experimental data are available in the literature to check the analysis in detail. A paper[14] describing model tests performed at the University of California presents no detailed information concerning the important characteristics of the vessel and its moorings, but by selecting a mooring with about the same characteristics in surge, one can obtain good agreement between experimental and calculated values of the response of an 880-ton vessel (Fig. 7). The calculation of the responses at a particular frequency is given in Appendix B.

The design of moorings by using the formulas of this Memorandum can be expedited considerably by graphical representation of the exciting forces and the impedances. Such a procedure is illustrated in Appendix B.

In practically all cases, the surge response of the vessel is the main contributor to high forces in the mooring lines. This is caused by the fact that very limited damping is available in this mode of movement.

In principle, a reduction of the surge response is possible by two methods: namely, by increasing the damping or by mismatching the natural frequency in surge with the main range of frequencies of wave excitations.

The application of the first method is limited because is is dif-

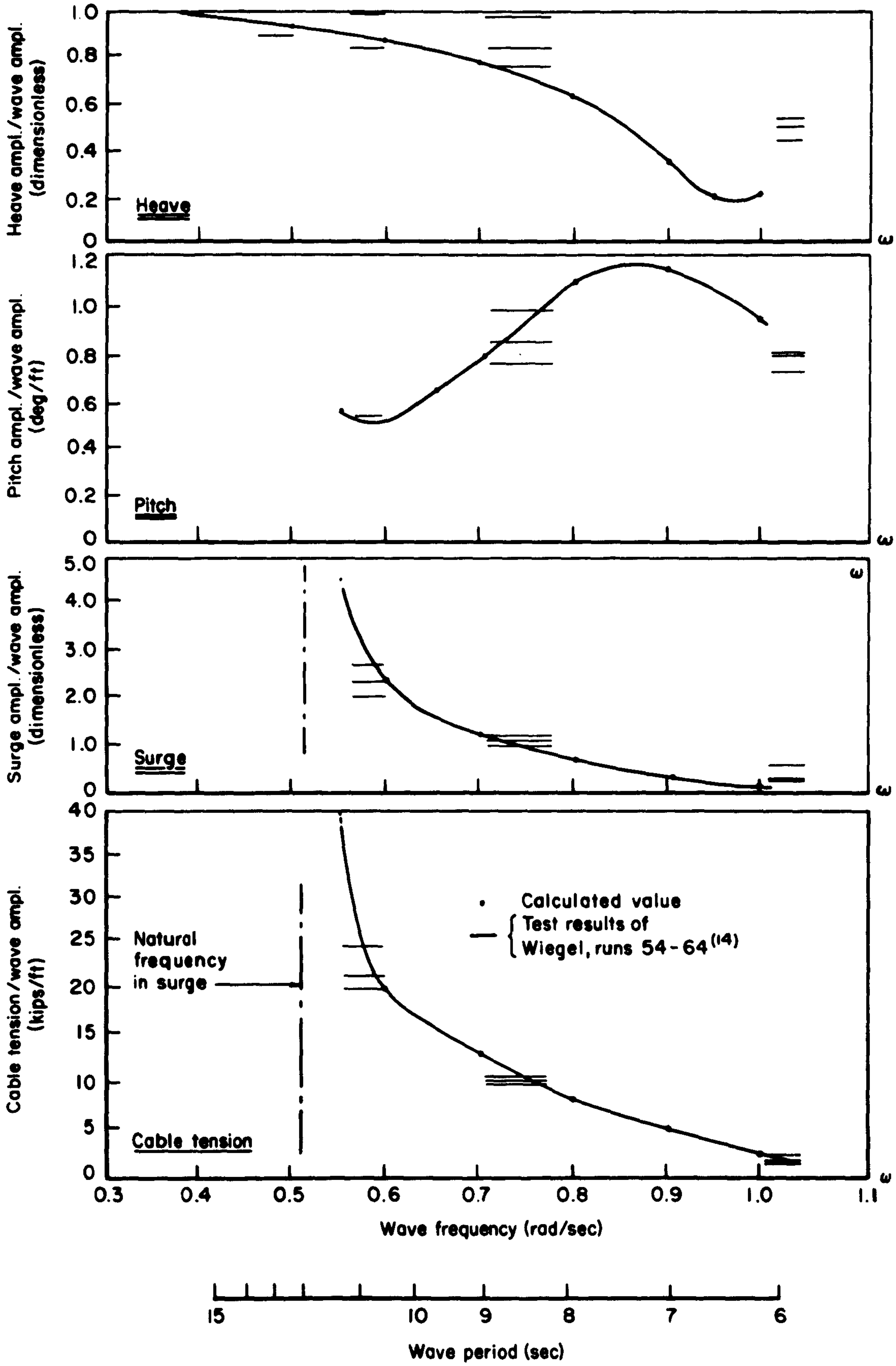


Fig. 7 — Experimental and calculated values of the response of an 880-ton vessel

ficult. In an incidental case, surge movements have been limited by the introduction of damping devices in the mooring lines.

Since wind and currents, whose effects are not discussed here, impose certain requirements on the mooring-line tensions, the applicability of the second method is often also limited.

In the previous sections of this Memorandum, a few relatively simple but realistic cases where the ship was moored in the longitudinal plane of symmetry were considered.

Unfortunately, the equations of motions in six degrees of freedom become very complicated, as coupling among many modes exists, and many of the hydrodynamic force coefficients cannot be calculated. Complete solutions of the motions are not yet realized.

Weinblum and St. Denis,[1] in their now classical paper, presented a method for calculating the uncoupled motions of an unrestrained ship in its six degrees of freedom in regular waves with arbitrary heading. This work has been expanded by Pierson and St. Denis[10] for the movement in irregular waves with a directional spectrum.

If the motions of the free-floating ship are considered uncoupled, the same ship in a moored condition will have coupled motions due to the mooring lines. For an arbitrary mooring, for example, the linearized equation of motion in surge becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{xy}y + K_{xz}z + K_{x\theta}\theta + K_{x\psi}\psi + K_{x\varphi}\varphi = \bar{\bar{F}}^{x}_{ex}\,Ae^{j\omega t} \tag{100}$$

The equations of motion in the other modes are similar. The coefficients K_{xy}, K_{xz}, etc., depend on the mooring lines and can be calculated in a manner similar to that for the spread-moored ship in waves head-on (e.g., Eqs. (6) through (8)).

For a submerged hovering submarine, a solution has been found by Goodman and Sargent.[7] It appeared that for an essentially symmetrical fore and aft body, the motions in the six degrees are uncoupled, with exception of the roll. The latter is coupled with yaw and sway. For the moored vessel, the surge and pitch become coupled in addition (Eqs. (77) and (78)).

Appendix A

MOORING-LINE CHARACTERISTICS

With reference to the symbols indicated in Fig. 8, the horizontal component (H) of the force in the line is constant, and the vertical component (V) is equal to the net weight of the section that is lifted off the bottom.

At an arbitrary point A, at distance s along the cable from the point where the cable touches the bottom

$$\frac{dy}{ds} = \frac{\bar{s}w/H}{\sqrt{1 + (\bar{s}w/H)^2}} \tag{101}$$

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + (\bar{s}w/H)^2}} \tag{102}$$

Integration over the length of the cable gives

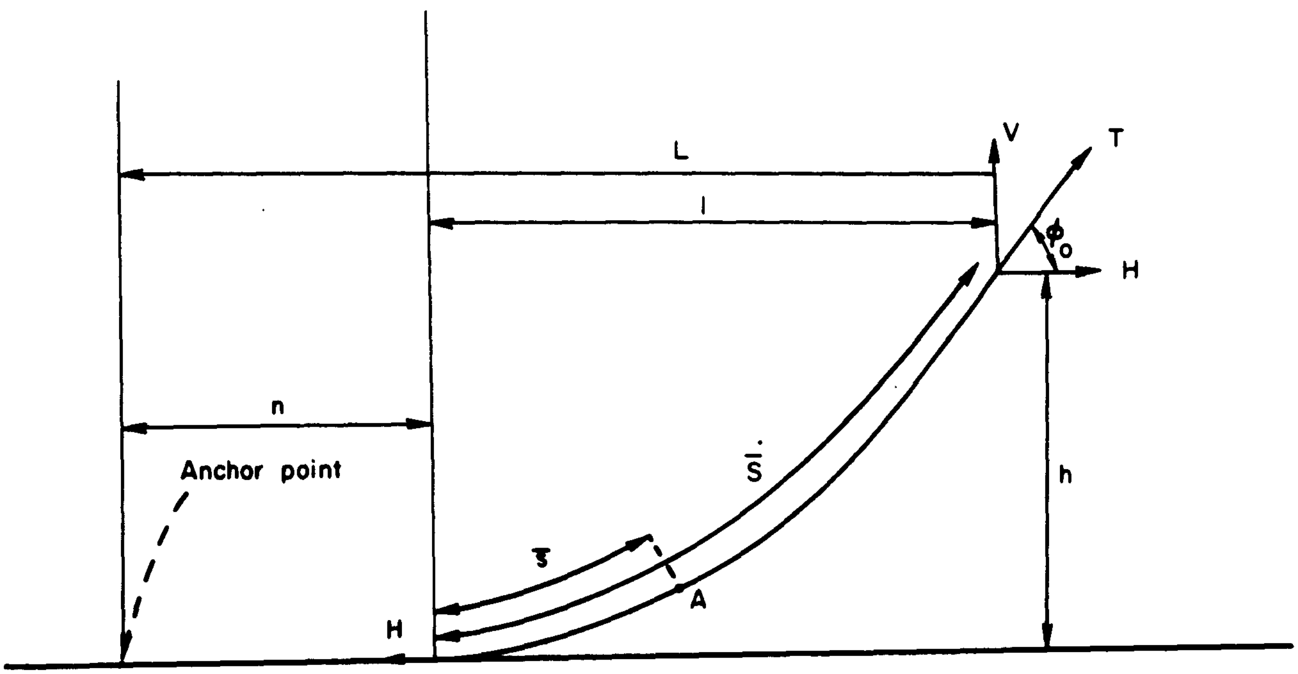
$$\ell = \int_0^{\bar{S}} \frac{d\bar{s}}{\sqrt{1 + (\bar{s}w/H)^2}} = \frac{H}{w}\left[\sinh^{-1}\frac{\bar{S}w}{H}\right] \tag{103}$$

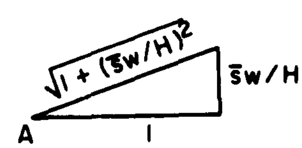
$$h = \int_0^{\bar{S}} \frac{\bar{s}w/H \; d\bar{s}}{\sqrt{1 + (\bar{s}w/H)^2}} = \frac{H}{w}\left[\sqrt{1 + \left(\frac{\bar{S}w}{H}\right)^2} - 1\right] \tag{104}$$

From Fig. 8 it can be established that

$$S - \bar{S} = n \tag{105}$$

$$L - \ell = n \tag{106}$$





w = weight per unit length of chain in water
S = total length of the cable
$\bar{S}$ = length of chain lifted from the bottom

Fig 8 — Schematic of mooring line and force components

thus

$$S - L = \bar{S} - \ell = \bar{S} - \frac{H}{w}\left[\sinh^{-1}\frac{\bar{S}w}{H}\right] \tag{107}$$

and

$$\frac{S - L}{h} = \frac{\bar{S}w/H - \sinh^{-1}\bar{S}w/H}{\sqrt{1 + \left(\frac{\bar{S}w}{H}\right)^2} - 1} \tag{108}$$

From Eq. (104)

$$\frac{H}{wh} = \frac{1}{\left[\sqrt{1 + \left(\frac{\bar{S}w}{H}\right)^2} - 1\right]} \tag{109}$$

consequently

$$\frac{V}{wh} = \frac{\bar{S}w/H}{\left[\sqrt{1 + \left(\frac{\bar{S}w}{H}\right)^2} - 1\right]} \tag{110}$$

$$\frac{T}{wh} = \frac{\sqrt{1 + \left(\frac{\bar{S}w}{H}\right)^2}}{\left[\sqrt{1 + \left(\frac{\bar{S}w}{H}\right)^2} - 1\right]} \tag{111}$$

and

$$\tan \phi_o = \frac{V}{H} = \frac{\bar{S}w}{H} \tag{112}$$

Equations (108) through (112) establish the relationship between the dimensionless parameters $\bar{S}w/H$, H/wh, V/wh, T/wh, $(S - L)/h$, and ϕ_o.

It will be noted in Eq. (108) that for a given value of $(S - L)/h$, $\frac{\bar{S}w}{H}$ can only be solved by trial and error.

Solutions are presented in (b) of Fig. 1. From these solutions the force variations, when the holding point is moved over a small distance in the horizontal or vertical direction according to Eqs. (2) and (3), can be calculated.

Appendix B

CALCULATION OF RESPONSES

The calculations of the responses, e.g., Eqs. (42), (43), and (47), are considerably simplified by introducing

$$\bar{f}^s_{ex} = \frac{\bar{F}^s_{ex}}{M_{ss}} \tag{113}$$

$$\kappa_{ss} = \frac{N_{ss}}{\omega_{o_s} M_{ss}} \tag{114}$$

$$\begin{aligned} \bar{z}_{ss} &= \frac{1}{\omega^2_{o_s}} \frac{\bar{Z}_{ss}}{M_{ss}} \\ &= 1 - \left(\frac{\omega}{\omega_{o_s}} \right)^2 + i \left(\frac{\omega}{\omega_{o_s}} \right) \kappa_{ss} \end{aligned} \tag{115}$$

$$k_{s\tau} = \frac{K_{s\tau}}{M_{ss}} \tag{116}$$

The value $\bar{z}_{ss}$ and its inverse $1/\bar{z}_{ss}$ are functions of the damping term κ_{ss} and the ratio of the exciting frequency and the natural frequency in the particular mode and are presented in Figs. 9 and 10.

The excitation term $\bar{f}^s_{ex}$ can also be plotted in the complex plane (for example, Fig. 11). Here $\bar{f}^x_{ex}$, $\bar{f}^z_{ex}$, and $\bar{f}^\theta_{ex}$ are plotted for a vessel of 880 tons with a length of 80 ft, draft of 5 ft, and beam of 33.6 ft, and a block coefficient $b_k = 1$.

This vessel has the same displacement, draft, and beam of a vessel studied by Wiegel.[14] The complex values of the excitation forces are calculated as outlined by Wilson,[4] which is a particular case

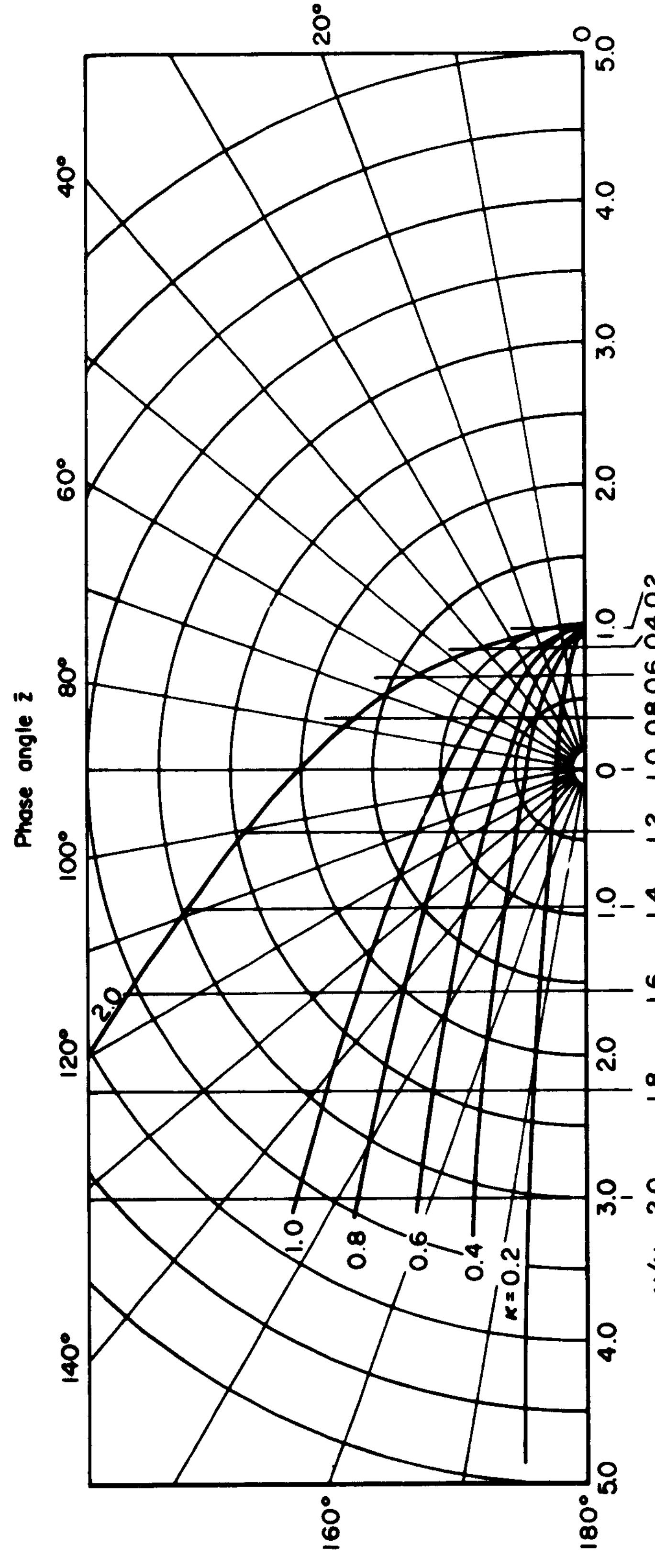


Fig. 9 — Polar plot of the impedance $\bar{Z}$

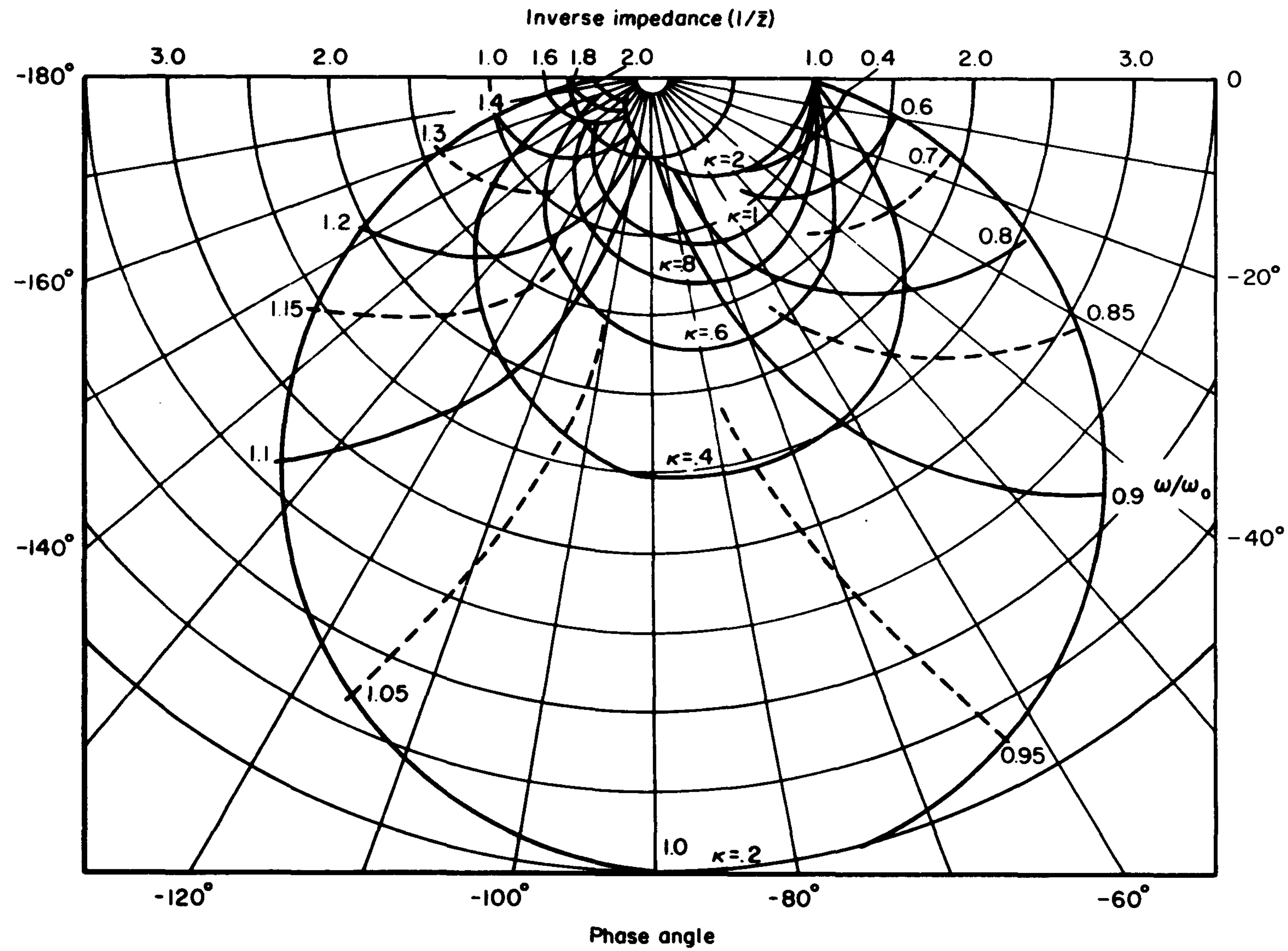


Fig. 10 —— Polar plot of the inverse impedance 1/Z

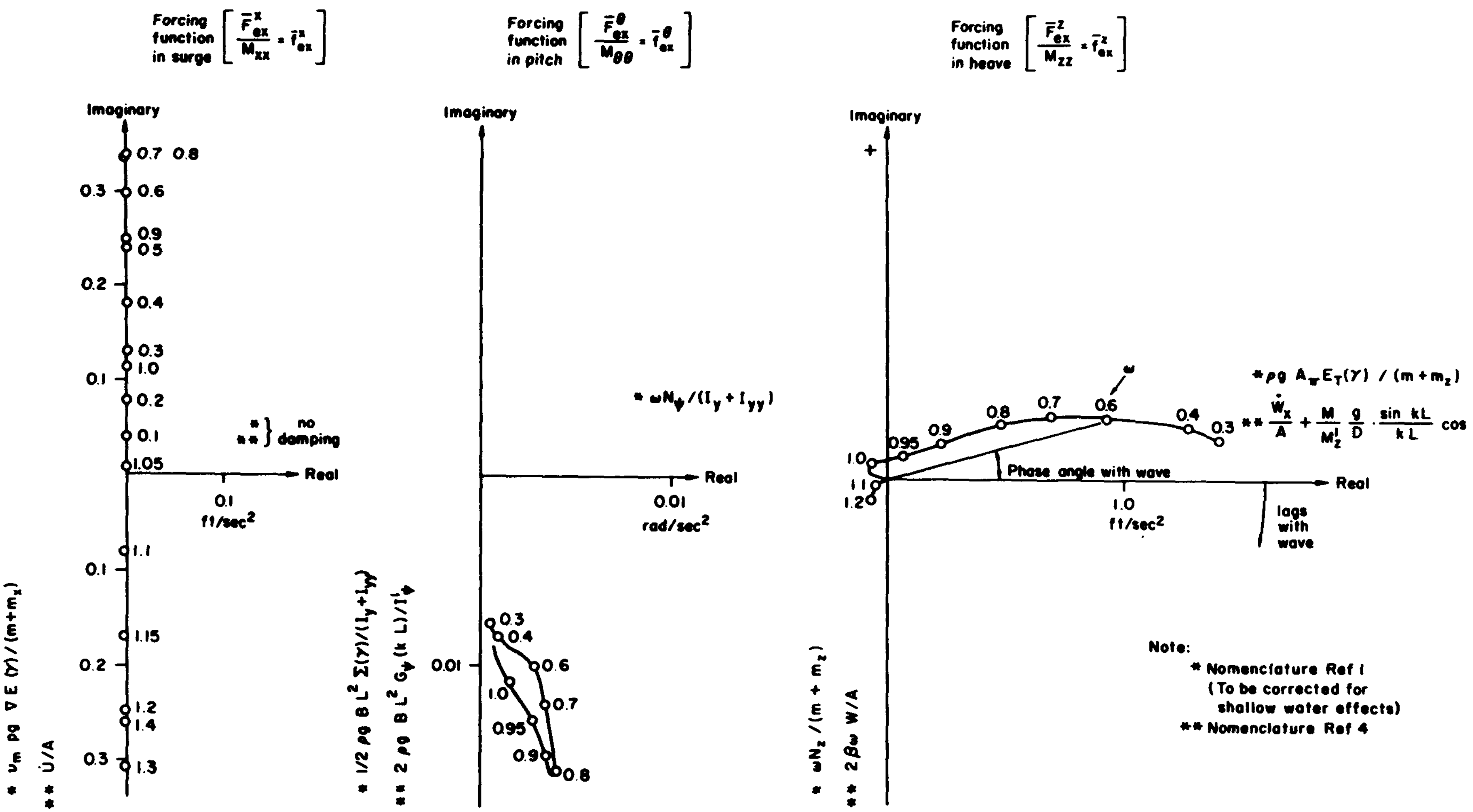


Fig. 11 — Complex representation of forcing functions in surge, pitch and heave as a function of frequency (ω) for an 880 ton vessel

of the method of Weinblum and St. Denis,[1] but corrections are made for the shallow-water effects.

The characteristics of the vessel are then chosen.

$$\left.\begin{aligned} M &= 60{,}400 \text{ slugs} \\ M''_x &= 9{,}000 \text{ slugs} \end{aligned}\right\} \quad M_{xx} = 69{,}400 \text{ slugs}$$

$$\left.\begin{aligned} I_\theta &= 250 \times 10^6 \text{ slugs-ft}^2 \\ I''_\theta &= 500 \times 10^6 \text{ slugs-ft}^2 \end{aligned}\right\} \quad M_{\theta\theta} = 750 \times 10^6 \text{ slugs-ft}^2$$

$$M''_z = 204{,}000 \text{ slugs}$$

$$M_{zz} = 264{,}400 \text{ slugs}$$

$$K_{\theta\theta} = J_Y = 1/12\ \rho g B(2L)^3 = 1040 \times 10^6 \text{ lb-ft/rad}$$

$$\kappa_{\theta\theta} = 0.4$$

$$\kappa_{zz} = 0.4$$

$$K_{zz} = \rho g(2LB) = 390{,}000 \text{ lb/ft}$$

$$\delta = 195 \text{ ft}$$

$$h = 200 \text{ ft}$$

Initial tension in bow and stern 1-1/2 in. die-lock chain ($w = 20$ lb/ft under water) $T = 90{,}000$ lb. Thus $T/wh = 22.5$, and from (b) of Fig. 1, $(S - L)/h = 0.1$. (Note $T \approx H$.)

The linear coefficients are obtained from Fig. 2.

$a/w = 450$		$a = 9000$ lb/ft	
$b/w = 45$		$b = 900$ lb/ft	
$c/w = 70$		$c = 1400$ lb/ft	
$d/w = 7$		$d = 140$ lb/ft	

$$K_{xx} = 2a = 18{,}000 \text{ lb/ft (Eq. 10)}$$

$$K_{x\theta} = -2(bL + ap) = -250{,}000 \text{ lb/ft (Eq. 11)}$$

$$K_{\theta x} = -2(cL + ap) = -340{,}000 \text{ lb/rad (Eq. 26)}$$

The natural frequencies are

$$\omega_{o_x} = \sqrt{K_{xx}/M_{xx}} = 0.51 \text{ rad/sec } (T_{o_x} = 2\pi/0.51 = 12 \text{ sec})$$

$$\omega_{o_\theta} = \sqrt{K_{\theta\theta}/M_{\theta\theta}} = 1.18 \text{ rad/sec } (T_{o_\theta} = 2\pi/1.18 = 5.3 \text{ sec})$$

$$\omega_{o_z} = \sqrt{K_{zz}/M_{zz}} = 1.21 \text{ rad/sec } (T_{o_z} = 2\pi/1.21 = 5.2 \text{ sec})$$

The uncoupled heave can be obtained directly from Figs. 10 and 11 and Eq. (47). For example, at $\omega = 0.6$

$$\bar{f}^z_{ex} = |0.96| \angle 15 \text{ deg}$$

$$\frac{1}{\bar{z}} = 1.30 \angle -13 \text{ deg}$$

$$\bar{C} = \frac{\bar{F}^z_{ex}}{\bar{Z}_{zz}} = \frac{\bar{f}^z_{ex}}{\omega^2_{o_z} \bar{z}_{zz}} A = \frac{0.96 \times 1.3}{1.44} A \angle 15 \text{ deg} - 13 \text{ deg}$$

$$\bar{C} = 0.87 A \angle 2 \text{ deg}$$

For surge from Eq. (42)

$$\bar{A} = \frac{\begin{vmatrix} \bar{f}^{x}_{ex} & k_{x\theta} \\ \bar{f}^{\theta}_{ex} & \omega^2_{o_\theta} \bar{z}_{\theta\theta} \end{vmatrix}}{\begin{vmatrix} \omega^2_{o_\theta} \bar{z}_{xx} & k_{x\theta} \\ k_{\theta x} & \omega^2_{o_\theta} \bar{z}_{\theta\theta} \end{vmatrix}} \quad A = \frac{\bar{f}^{x}_{ex} \cdot \omega^2_{o_\theta} \bar{z}_{\theta\theta} - k_{x\theta} \bar{f}^{\theta}_{ex}}{\omega^2_{o_x} \bar{z}_{xx} \, \omega^2_{o_\theta} \bar{z}_{\theta\theta} - k_{x\theta} \, k_{\theta x}}$$

where

$$\bar{f}^{x}_{ex} = 0.3 \angle 90 \text{ deg}$$

$$\bar{f}^{\theta}_{ex} = 10.5 \times 10^{-3} \angle -74 \text{ deg}$$

$$k_{x\theta} = -250{,}000/69{,}400 = -3.61$$

$$k_{\theta x} = -340{,}000/750 \times 10^{6} = -0.455 \times 10^{-3}$$

$$\omega_{o_\theta} = 1.18 \text{ rad/sec}, \qquad \frac{\omega}{\omega_{o_\theta}} = \frac{0.6}{1.18} = 0.508$$

$$\bar{z}_{\theta\theta} = 0.76 \angle 14 \text{ deg}, \qquad \omega^2_{o_\theta} \bar{z}_{\theta\theta} = 1.06 \angle 14 \text{ deg}$$

$$\omega_{o_x} = 0.51 \text{ rad/sec}, \qquad \frac{\omega}{\omega_{o_x}} = \frac{0.6}{0.51} = 1.18$$

$$\bar{z}_{xx} = 0.4 \angle 180 \text{ deg}, \qquad \omega^2_{o_x} \bar{z}_{zz} = 0.1 \angle 180 \text{ deg}$$

and

$$\left.\begin{aligned} \bar{f}^{x}_{ex} \, \omega^2_{o_\theta} \bar{z}_{\theta\theta} &= 0.318 \angle 104 \text{ deg} \\ -k_{x\theta} \, \bar{f}^{\theta}_{ex} &= 38.0 \times 10^{-3} \angle -74 \text{ deg} \end{aligned}\right\} \text{Graphical addition results in } 0.280 \angle 104 \text{ deg}$$

$$\left.\begin{aligned} \omega_{o_x}^2 \bar{z}_{xx}\, \omega_{o_\theta}^2 z_{\theta\theta} &= 0.106 \angle 194 \text{ deg} \\ -k_{x\theta}\, k_{\theta x} &= -1.64 \times 10^{-3} \end{aligned}\right\} \text{Graphical addition results in } 0.12 \angle 193 \text{ deg}$$

$$\bar{A} = \frac{0.280 \angle 104 \text{ deg}}{0.12 \angle 193 \text{ deg}} A = 2.33\, A \angle -89 \text{ deg}$$

For the pitch

$$\bar{B} = \frac{\begin{vmatrix} \omega_{o_x}^2 \bar{z}_{xx} & \bar{f}_{ex}^{x} \\ k_{\theta x} & \bar{f}_{ex}^{\theta} \end{vmatrix}}{\begin{vmatrix} \omega_{o_x}^2 \bar{z}_{xx} & k_{x\theta} \\ k_{\theta x} & \omega_{o_\theta}^2 \bar{z}_{\theta\theta} \end{vmatrix}} = \frac{\bar{f}_{ex}^{\theta}\, \omega_{o_x}^2 \bar{z}_{xx} - \bar{f}_{ex}^{x}\, k_{\theta x}}{\omega_{o_x}^2 \bar{z}_{xx}\, \omega_{o_\theta}^2 \bar{z}_{\theta\theta} - k_{x\theta}\, k_{\theta x}}$$

$$\left.\begin{aligned} \bar{f}_{ex}^{\theta}\, \omega_{o_x}^2 z_{xx} &= 1.05 \times 10^{-3} \angle 106 \text{ deg} \\ -\bar{f}_{ex}^{x}\, k_{\theta x} &= +0.136 \times 10^{-3} \angle 90 \text{ deg} \end{aligned}\right\} \text{Graphical addition results in } 1.15 \times 10^{-3} \angle 104 \text{ deg}$$

$$\bar{B} = \frac{1.15 \times 10^{-3} \angle 104 \text{ deg}}{0.12 \quad \angle 193 \text{ deg}} = \begin{cases} 0.95 \times 10^{-2} \text{ A rad} \angle -89 \text{ deg} \\ 0.54 \text{ A deg} \angle -89 \text{ deg} \end{cases}$$

The force variation in the stern chain can be calculated

$$\begin{aligned} T_{stern}(A, 0.6) = H_{stern}(A, 0.6) &= \left[-aA + (ap + bL)\bar{B} + b\bar{C}\right]A \\ &= \Big[(-21{,}000 \angle -89 \text{ deg}) + (1200 \angle -89 \text{ deg}) \\ &\quad + (780 \angle 2 \text{ deg})\Big]A \end{aligned}$$

which, by solving graphically, results in

$$T_{stern}(A, 0.6) = \left[20{,}000 \angle 89 \text{ deg}\right] A\, e^{j\omega t}$$

It will be noticed that force variation is caused mainly by the surge.

The results of the responses in heave, surge, and pitch, as to phase angle and amplitude, are in good agreement with the results of a model test (Fig. 12) in waves with comparable frequency. The characteristics of the cables could not be determined from the paper, but the natural frequency in surge was approximately the same ($\omega_{model_x} = 0.4 - 0.5$; $\omega_{anal_x} = 0.51$).

Figure 6 presents also the calculated responses in heave, pitch, and surge and the tension in the stern cable for a range of frequencies. The results are in good agreement with experimental values.

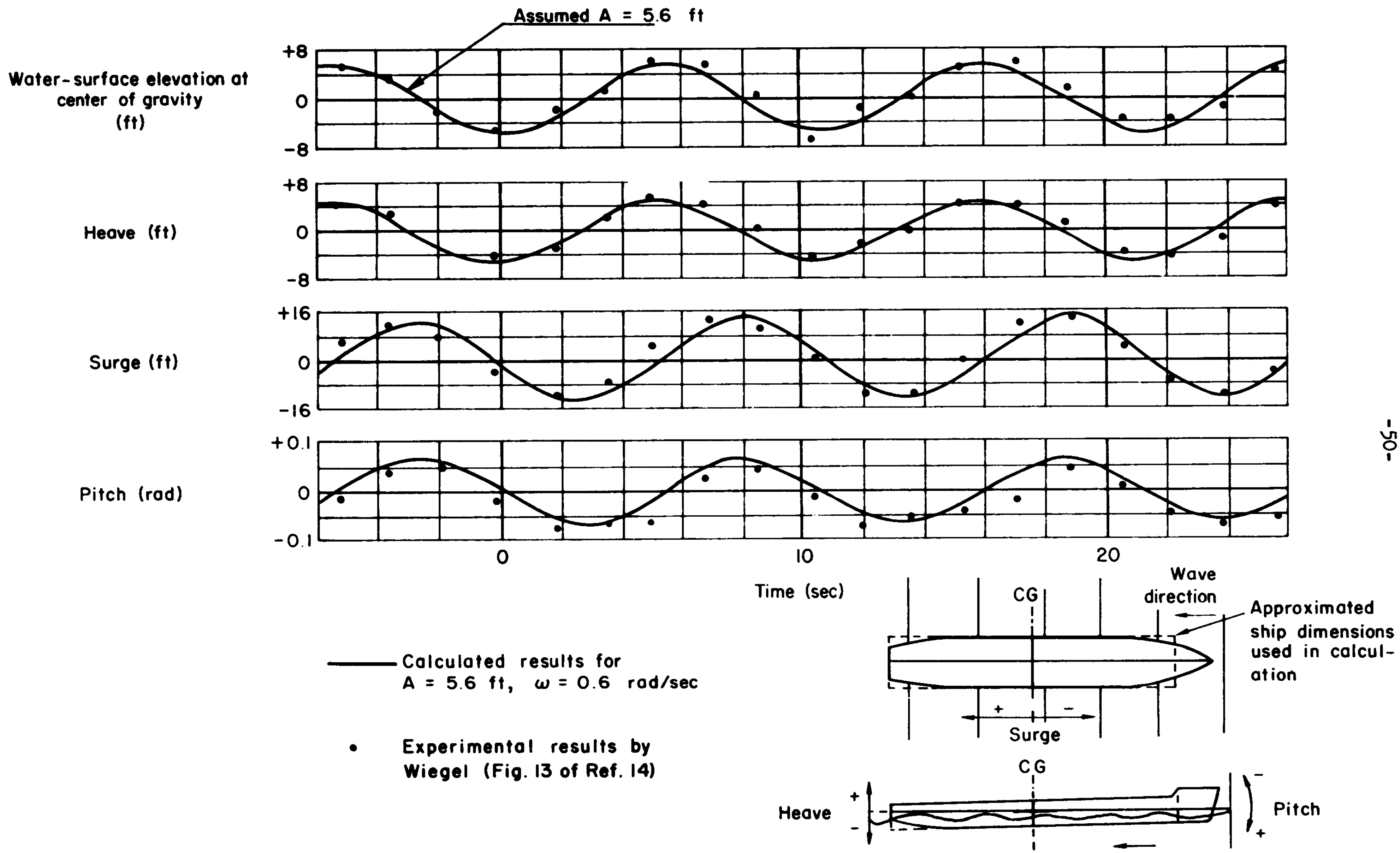


Fig. 12 — Comparison between calculated and measured responses of a moored ship in uniform waves

REFERENCES

1. Weinblum, Georg, and Manley St. Denis, "On the Motions of Ships at Sea, Trans. SNAME, Vol. 58, 1950, pp. 184-231.

2. Korvin-Kroukovsky, B. V., Theory of Seakeeping, Society of Naval Architects and Marine Engineers, New York, 1961.

3. Korvin-Kroukovsky, B. V., and Winnifred R. Jacobs, "Pitching and Heaving Motions of a Ship in Regular Waves," Trans. SNAME, Vol. 65, 1957, pp. 590-632.

4. Wilson, Dr. Basil W., "The Energy Problem in the Mooring of Ships Exposed to Waves," Perm. Int. Assoc. of Nav. Congresses Bull. No. 50, 1959.

5. Kriloff, A., "A General Theory of the Oscillations of a Ship on Waves," and "On Stresses Experienced by a Ship in a Seaway," INA, Vol. 40, 1898, pp. 135-212.

6. Weinblum, Georg, "Progress of Theoretical Investigations of Ship Motions in a Seaway," Ships and Waves, 1954, pp. 129-159.

7. Goodman, Theodore R., and Theodore P. Sargent, "Launching of Airborne Missiles Underwater," Part IX, Allied Research Assoc., Inc., Boston, Mass., (ASTIA AD 251 620), 1961.

8. Pierson, Williard J., Gerhard Neumann, and Richard W. James, Observing and Forecasting Ocean Waves, Hydrographic Office Pub. 603, Reprinted 1960.

9. Wiegel, R. L., and J. W. Johnson, "Elements of Wave Theory," Proc. of the First Conf. on Coastal Engineering, Council on Wave Research, Berkeley, California, 1951.

10. St. Denis, Manley, and Willard J. Pierson, Jr., "On the Motions of Ships in Confused Seas," Trans. SNAME, Vol. 61, 1953, pp. 280-357.

11. Longuet-Higgins, M. S., "On the Statistical Distribution of the Heights of Sea Waves," J. Mar. Res., Vol. 11, 1952, pp. 245-266.

12. Bendat, Julius S., Principles and Applications of Random Noise Theory, John Wiley and Sons, New York, 1958, pp. 130-133.

13. Crandall, Stephen, Random Vibrations of Systems with Non-Linear Restoring Forces, Massachusetts Institute of Technology, AFOSR 708 (ASTIA No. AD 259693), June 1961.

14. Wiegel, R. L., "Model Studies of the Dynamics of an LSM Moored in Waves," Proc. of Sixth Conf. on Coastal Engineering, Council on Wave Research, Berkeley, California, 1958.